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Anisotropic elastodynamics in a half space: An analytic method for polynomial data☆

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Abstract

An initial boundary value problem for the dynamic system of anisotropic elasticity in a half space is studied in the paper. A novel method of finding an exact solution of this problem for a special polynomial form of initial data and inhomogeneous term of the system is described. On the base of this method the simulation of elastic waves in different homogeneous anisotropic half spaces is implemented.

Keywords: Anisotropic elasticity; Dynamic system; Initial boundary value problem; Analytic method; Simulation

1. Introduction

A mathematical model of wave propagations in anisotropic elastic materials is described by the dynamic system of anisotropic elasticity which usually has been studied by the plane wave approach [4,8]. However, nowadays there is a great interest to develop new methods for solving initial value problems (IVPs) and initial boundary value problems (IBVPs) for the dynamic system of anisotropic elasticity and simulate invisible elastic waves [1,2,9]. Most of the time the numerical methods, in particular the finite element method, are used for solving this kind of problems. Advantages and disadvantages of these methods are well known [2,10]. Generally speaking, they are of a general purpose, rather labor-consuming, find approximate solutions, but do not always satisfy scientists and engineers at the needed scale and accuracy [6]. At the same time analytic methods can provide the exact solution of the equations and also offer a fundamental understanding of the relevant physical phenomena. Unfortunately the exact solutions cannot be found for all complex equations and systems. But when the exact solutions can be found it leads to a significant simplification of modeling and simulation. The modern methods of symbolic computations allow us to automate mathematical transformations on a very high level of complexity thanks to the truly remarkable achievements in computing power over the last decade [6].

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In our paper we present a new analytic method for constructing an exact solution of IBVP for the system of elastodynamics with polynomial data describing wave propagations in a homogeneous anisotropic half space with the general structure of anisotropy. This method is based on the following. Using assumptions that data and the inhomogeneous term of the elastic system have a special polynomial form with respect to lateral variables we seek a solution of IBVP in the same polynomial form with unknown function-coefficients depending on time and vertical variables. These function-coefficients are found successively by a recurrence procedure as explicit formulae. Unfortunately these formulae are very cumbersome and it is almost impossible to calculate them "by hands". At this point the symbolic computations help us to get these explicit formulae. By means of symbolic computations in Maple the explicit formulae of solutions of IBVP for different types of anisotropy and polynomial data were obtained. Using these formulae we have simulated elastic wave propagations in different anisotropic half spaces to confirm the robustness of the suggested method.

The paper is organized as follows. In Section 2 we state the IBVP for the system of anisotropic elastodynamics in a half space and specify assumptions. In Section 3 we describe a procedure of finding a vector-function and its properties. We show here that this constructed vector-function is a generalized (weak) solution of IBVP and this solution is unique inside of a bounded domain of dependence. Section 4 contains examples of elastic fields computations and their simulations. The simulations are presented by figures showing the dynamics of elastic waves in different anisotropic solids.

2. Statement of the problem

2.1. Problem setup

The IBVP of anisotropic elastodynamics in an elastic half space has the following form:

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} \frac{\partial}{\partial x_k} \left(c_{jklm} \frac{\partial u_l}{\partial x_m} \right) + f_j(x,t), \tag{1}$$

$$u_j(x,0) = \varphi_j(x), \qquad \left. \frac{\partial u_j(x,t)}{\partial t} \right|_{t=0} = \psi_j(x), \tag{2}$$

$$\sum_{l=1}^{3} \sum_{m=1}^{3} c_{j3lm} \left. \frac{\partial u_l}{\partial x_m} \right|_{x_3=0} = 0, \quad j = 1, 2, 3, \tag{3}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^2 \times (0, \infty)$, $t > 0$, $u_j(x,t)$ is the jth component of the displacement vector $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$, ρ is the density of the elastic medium; $\{c_{jklm}\}_{j,k,l,m=1}^{3}$ are the elastic moduli of the medium.

The main problem of this paper is to find unknown functions $u_j(x,t)$, $j = 1, 2, 3$ which are a solution of (1)–(3) if functions $\varphi_j(x)$, $\psi_j(x)$, $f_j(x,t)$, $j = 1, 2, 3$ are given for $x = (x_1, x_2, x_3) \in \mathbb{R}^2 \times [0, \infty)$, $t \in [0, \infty)$.

2.2. Assumptions

The natural assumptions for elastic moduli c_{jklm} and density ρ are the following [4]. The density ρ and elastic moduli c_{jklm} are constants, $\rho > 0$, c_{jklm} satisfy the following symmetry properties $c_{jklm} = c_{lmjk} = c_{kjlm}$ and the elastic tensor $\{c_{jklm}\}_{j,k,l,m=1}^{3}$ is positive definite. This means that there exists a positive constant M such that $\sum_{j,k,l,m=1}^{3} c_{jklm} \varepsilon_{jk} \varepsilon_{lm} \geqslant M \sum_{j,k}^{3} \varepsilon_{jk}^2$ for all ε_{jk} such that $\varepsilon_{jk} = \varepsilon_{kj}$. The elastic moduli can be represented by a 6×6