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A stable well-conditioned integral equation for electromagnetism scattering

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Abstract

Finding a formulation for electromagnetic scattering of surfaces which is both well-posed and produces a well-conditioned linear system is still a challenging problem. We here propose one such formulation valid in the high-frequency regime. The mathematical analysis is provided and numerical results on rather complex geometries show the performance of the method.

Keywords: Time-harmonic scattering problem; Maxwell equations; Iterative method; Integral equation; Helmholtz decomposition; CFIE; Preconditioner

1. Introduction

The use of integral equations to solve wave scattering has become very popular from the sixties since the method intrinsically permits to reduce by one the dimension of the problem (e.g., to pose the problem on a scattering surface only instead of the whole space). However, it appeared that several formulations, interesting at first sight because of their simplicity of discretization or their physical meaning, are actually ill-posed for some (bad) frequencies. Adapting to acoustics ideas that were developed in the context of elasticity by the Russian school (mainly represented by Kupradze [14]), Brackhage and Werner [6] and Panich [22] observed that a linear combination of equations which do not possess the same irregular frequencies may (provided the coupling coefficients are well chosen) produce a non-resonant equation. When applied to electromagnetism, this technique will be very fruitful to write well-posed equations at all frequencies. Maybe the two most famous ones are those developed by Mitzner [21] which is a combined field integral equation (CFIE) and the one by Mautz and Harrington [19] which is a combined source integral equations (CSIE). Although both equations are based on common principles, the CFIE will be the most successful. Easier to implement, and more natural from a physical viewpoint (maybe also more precise than its counterpart), the CFIE belongs now to the family of classical equations, and most of the industrial codes devoted to electromagnetism solve it.

Nonetheless, although stable equations exist, when people turned to solve bigger and bigger cases (obtained by finer and finer discretization), the use of direct solvers became impossible, and the switching to iterative methods (the development of GMRES [25] brought a substantial improvement in that direction) posed both the question of finding a

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fast matrix–vector multiplication, and the question of speeding up the convergence of the iterative method. The former problem is now considered to be solved by using the fast multipole method (FMM) of Rokhlin (see [24] and later on the papers by Greengard and Rokhlin [12] for instance) which reduces the $\mathrm{O}(N^2)$ classical algorithm for matrix–vector multiplication to a $\mathrm{O}(N\log(N))$ complexity for such problems, and allows now, on modern computers, to consider problems with a number of unknowns exceeding the million.

The remaining question is those of convergence speed of the iterative method (GMRES for instance) to solve the linear system coming from the discretization of the problem. It is well-known that it is directly related to the condition number of the matrix, and it turned out that though well-posed for all frequencies the best formulations (like the CFIE) may produce a very ill-conditioned linear system for which the numerical resolution by means of iterative methods becomes costly. The traditional cure for this problem is to use a preconditioner to the linear system, which is roughly equivalent to multiply the original system by a matrix such that the resulting matrix is close to identity. However, such preconditioners are not easy to compute as their pure algebraic nature may not easily take into account the dependence in the frequency. The efficiency of these preconditioners is moreover difficult to analyze theoretically and sometimes also not quite convincing from a practical point of view.

The following idea (though for different problems [20,26,27]) was to stabilize the formulation not after their discretization (after the assembling of the linear system), but at the very beginning of the conception of the integral equation, by finding a parametrix of the underlying operator. We take this point of view here in the framework of scattering. As it has been pointed out by several authors [8,10], though conceptually clear, this program is not easy to concretize in practice. This paper is devoted to give an example of such a strategy. Indeed, Section 3 shows how to construct a new intrinsically well-conditioned integral equation which leads after discretization to linear systems for which classical iterative methods converge quickly without the need of any preconditioner. Section 2 makes a (small) review of integral equations in sources versus in fields, where it is shown that for our objectives the source formulations may be more suitable. The discretization is treated in Section 4 and numerical results demonstrating the robustness and real applicability of the method are provided in Section 5.

2. Integral equations in sources and in fields

In this section, we recall the model problem on which we work and the classical integral equations usually used to solve it. We will focus on two different strategies: the CSIEs and the CFIEs.

Let Ω be a three-dimensional bounded domain with a smooth boundary Γ. We call $\mathbf{n}$ the outward unit normal on Γ. We define W^+ as the space of radiating electric fields $\mathbf{E}$ solutions of Maxwell equations in $\mathbb{R}^3\backslash\overline{\Omega}$ which have a tangential trace on Γ. Our problem writes as follows:

$$\text{Find } \mathbf{E}\in W^+ \text{ such that } \mathbf{n}\times\mathbf{E}=-\mathbf{n}\times\mathbf{E}^{\mathrm{inc}} \quad \text{on } \Gamma, \tag{1}$$

which models the scattering by a perfectly conducting material of an incident wave $\mathbf{E}^{\mathrm{inc}}$.

A rather natural way to solve (1) with an integral equation, consists in giving us a parameterization of admissible fields W^+ with a functional which links current distributions on Γ and electric fields of W^+

$$\mathscr{V}:\mathscr{D}'_T(\Gamma)\to W^+, \tag{2}$$

where $\mathscr{D}'_T(\Gamma)$ is the space of tangential vectorial distributions on Γ. The corresponding integral equation becomes

$$\mathbf{n}\times\mathscr{V}(\mathbf{u})=-\mathbf{n}\times\mathbf{E}^{\mathrm{inc}} \quad \text{on } \Gamma, \tag{3}$$

where the unknown $\mathbf{u}$ is not necessarily physically meaningful (in other words, $\mathbf{u}$ does not need to be a Cauchy data of the solution $\mathbf{E}$ to (1)). Classical potentials in $\mathbb{R}^3\backslash\Gamma$ are given by

$$\mathscr{L}=\frac{1}{\mathrm{i}k}\nabla\times\nabla\times\mathscr{G} \quad \text{and} \quad \mathscr{K}=\nabla\times\mathscr{G}, \tag{4}$$

where $\mathscr{G}$ stands for the vector potential (which depends on the wavenumber k) which to a tangent vectorfield $\mathbf{u}$ on Γ associates the vector-field defined on $\mathbb{R}^3\backslash\Gamma$ by

$$\mathscr{G}\mathbf{u}(x)=-\frac{1}{4\pi}\int_\Gamma \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|}\mathbf{u}(y)\,\mathrm{d}y. \tag{5}$$