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General recurrence and ladder relations of hypergeometric-type functions☆

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Abstract

A method for the explicit construction of general linear sum rules involving hypergeometric-type functions and their derivatives of any order is developed. This method only requires the knowledge of the coefficients of the differential equation that they satisfy, namely the hypergeometric-type differential equation. Special attention is paid to the differential-recurrence or ladder relations and to the fundamental three-term recurrence formulas. Most recurrence and ladder relations published in the literature for numerous special functions including the classical orthogonal polynomials, are instances of these sum rules. Moreover, an extension of the method to the generalized hypergeometric-type functions is also described, allowing us to obtain explicit ladder operators for the radial wave functions of multidimensional hydrogen-like atoms, where the varying parameter is the dimensionality.

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1. Introduction

The study of linear and nonlinear N-term ($N \geqslant 3$) relations among the special functions of mathematical physics and their derivatives of any order is a very relevant mathematical problem from both fundamental [3,8,12–14,19,20,23,26,28,41,43,44] and applied [2,4,9,11,29,31,32,46] standpoints. If $\{y_\alpha(z)\}_{\alpha\in\Lambda}$ denotes a specific one-parameter family of special functions, being α the parameter running over a certain set of indices Λ, a wide and important subset of such algebraic properties can be described by the general sum rule,

$$\sum_{i=0}^{N-1} A_{\alpha_i}(z)\frac{\mathrm{d}^{k_i}}{\mathrm{d}z^{k_i}}[y_{\alpha_i}(z)] = 0 \quad (N \geqslant 3,\ \alpha_i \in \Lambda,\ i = 0, 1, \ldots, N-1), \tag{1}$$

☆ This paper has been written to honor Nico Temme on occasion of his 65th birthday.

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linearly linking $N \geqslant 3$ different functions and/or their derivatives of arbitrary order. Some interesting particular instances of Eq. (1) are the fundamental three-term recurrence formula ($N = 3$, $k_i = 0$, $\alpha_i = \alpha + i$) and the ladder relations (also called structure relations [3,12] in the theory of orthogonal polynomials), which allows us to find ladder operators (i.e., first-order differential operators shifting the index α by ± 1, i.e., $N = 3$, $k_0 = 1$, $k_1 = k_2 = 0$, $\alpha_0 = \alpha_1 = \alpha$, $\alpha_2 = \alpha \pm 1$ in Eq. (1)).

The study of these sum rules include the two following problems: to find existence conditions of these relationships and to compute explicitly the coefficients $A_{\alpha_i}(z)$ $(i = 0, 1, \ldots, N-1)$ which completely characterize the corresponding algebraic and/or differential properties as given by Eq. (1). For solving these problems some general methods have been designed in the literature which use as starting point some specific property of the involved special functions. Among them, let us mention the early works of Inoui [26] and Truesdell [41], and those of Barik [8] and Hansen [23] where the starting point is a recurrence formula that the functions must satisfy as well as generating function technique developed by Dattoli et al. [13–17]. More specific is the method of Marcellán et al. [28], which provides an unified way of obtaining the three-term recurrence and structure relations for the classical orthogonal polynomials starting from the Pearson-type equation satisfied by the corresponding "orthogonalizing" weight function. Some sum rules could be also obtained when the function admits a representation as a generalized hypergeometric series [1,22,37].

Special attention has been paid to the construction of ladder operators because of the important role they play on solving the fundamental wave equation in relativistic and nonrelativistic quantum mechanics (see [27] for an state-of-the-art of this subject up to the nineties). In this context those operators are usually called "creation" and "annihilation" operators and their significance comes from a Dirac idea [21], which was developed later on, giving rise to the well known "factorization method" [25], for which a second-order differential equation satisfied by the involved functions is required. Also, some alternative approaches for computing recurrences have been devised, not only for solving the wave equation, but also for the computation of matrix elements, mainly referring to hydrogenic and oscillator-like wave functions (see e.g., [9,31,32]).

Let us consider the *generalized hypergeometric-type differential equation* [34], that is the equation

$$u''(z) + \frac{p(z)}{Q(z)}\, u'(z) + \frac{q(z)}{[Q(z)]^2}\, u(z) = 0, \tag{2}$$

where $q(z)$ and $Q(z)$ are polynomials of degree at most two, and $p(z)$ is also a polynomial of degree at most one. Solutions of Eq. (2) form an important subset of special functions usually known (in a wide sense) as "special functions of mathematical physics" because of their occurrence in many fundamental problems of applied mathematics and mathematical physics [30,34,5,6,40]. A way to study this equation begins with a set of changes of the independent variable [34] which transforms Eq. (2) into the simpler equation

$$\sigma(z)\, y''(z) + \tau(z) y'(z) + \lambda y(z) = 0, \tag{3}$$

called by *hypergeometric-type differential equation*, where $\sigma(z)$ and $\tau(z)$ are polynomials of degree not greater than two and one, respectively, and λ is a constant. In fact, any linear second-order differential equation whose singularities are regular and no more than three in number, can be transformed into an equation of this form [35, Theorem 8.1].

In the late eighties Nikiforov and Uvarov proposed a method [34, p. 14], later on developed and extended [20,43–46], to compute algebraic and differential properties of the type (1) for *hypergeometric-type functions*, which are some solutions of Eq. (3). Under appropriate conditions (see [34, p. 14] and [20,43–46]) this method allows us an unified computation of coefficients $A_{\alpha_i}(z)$ in Eq. (1) directly in terms of the generic coefficients $\sigma(z)$ and $\tau(z)$ of the differential equation. This method is based upon the use of an integral representation [34, p. 9] for the solutions of Eq. (3); see also Ref. [43] for a detailed description. Therein, we realize that this method applies to hypergeometric-type equations of the type (3) in which the coefficients $\sigma(z)$ and $\tau(z)$ do not depend on the spectral parameter λ.

The main purpose of this work is to extend the Nikiforov–Uvarov method to the hypergeometric-type differential equation (3) where the coefficients (σ, τ, λ) are parameter dependent, i.e.,

$$\sigma(z; \bar{c})\, y'' + \tau(z; \bar{c})\, y' + \lambda(\bar{c})\, y = 0, \tag{4}$$

where $\bar{c}$ stands for a set of meaningful parameters, being 3 the irreducible number of them, so that the σ and τ coefficients change when the spectral parameter λ does so. This kind of equation are naturally encountered from the generalized