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Theory and numerical evaluation of oddoids and evenoids: Oscillatory cuspoid integrals with odd and even polynomial phase functions

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Abstract

The properties of oscillating cuspoid integrals whose phase functions are odd and even polynomials are investigated. These integrals are called oddoids and evenoids, respectively (and collectively, oddenoids). We have studied in detail oddenoids whose phase functions contain up to three real parameters. For each oddenoid, we have obtained its Maclaurin series representation and investigated its relation to Airy–Hardy integrals and Bessel functions of fractional orders. We have used techniques from singularity theory to characterise the caustic (or bifurcation set) associated with each oddenoid, including the occurrence of complex whiskers. Plots and short tables of numerical values for the oddenoids are presented. The numerical calculations used the software package CUSPINT [N.P. Kirk, J.N.L. Connor, C.A. Hobbs, An adaptive contour code for the numerical evaluation of the oscillatory cuspoid canonical integrals and their derivatives, Comput. Phys. Commun. 132 (2000) 142–165].

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1. Introduction

This paper is the third in a series [28,29] concerned with the numerical evaluation and properties of oscillating integrals. In our first paper [29], we described a FORTRAN 90 code, called CUSPINT, which was written for the numerical computation by quadrature of the cuspoid canonical integrals

$$C_n(\boldsymbol{\alpha}) = \int_{-\infty}^{\infty} \exp\left[\mathrm{i}\left(u^n + \sum_{j=1}^{n-2} \alpha_j u^j\right)\right] \mathrm{d}u, \quad n = 3, 4, 5, \ldots \tag{1}$$

and their partial derivatives, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{n-2})$ is a vector of real numbers.

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CUSPINT implements a novel adaptive algorithm, which chooses contours in the complex u plane that avoid the violent oscillatory and exponential natures of the integrand and modifies its choice as necessary [29]. This adaptive contour algorithm has the advantage that it is relatively easy to implement on a computer, is efficient, and provides highly accurate results.

Our second paper [28] showed how a modified version of CUSPINT could be used for the numerical evaluation of other types of oscillating integrals; in particular we studied the bessoid canonical integral

$$J(x, y) = \int_0^\infty J_0(yu)u \exp[\mathrm{i}(u^4 + xu^2)]\, \mathrm{d}u, \tag{2}$$

where $J_0(\bullet)$ denotes the Bessel function of order zero. The bessoid integral arises in the theory of axially symmetric cusped focusing [3,28,30,38,40, p. 402]. In Ref. [18], two of us (CAH and JNLC) have provided an overview of the research reported in [28,29]. Gil et al. [22] have emphasised recently that quadrature methods are of great importance for the evaluation of special functions.

The purpose of this paper is to investigate the properties of two classes of oscillating integrals, which we will call *oddoids* and *evenoids*. The (real) oddoid integrals of order $k = 1, 2, 3, \ldots$ are defined by

$$\begin{aligned} O_k(\mathbf{a}) &= \int_{-\infty}^\infty \exp\left[\mathrm{i}\left(\frac{t^{2k+1}}{2k+1} + \sum_{j=1}^k a_j \frac{t^{2j-1}}{2j-1}\right)\right] \mathrm{d}t \\ &= 2\int_0^\infty \cos\left(\frac{t^{2k+1}}{2k+1} + \sum_{j=1}^k a_j \frac{t^{2j-1}}{2j-1}\right) \mathrm{d}t, \end{aligned} \tag{3}$$

whereas the (complex) evenoid integrals of order $k = 1, 2, 3, \ldots$ are defined by

$$\begin{aligned} E_k(\mathbf{a}) &= \int_{-\infty}^\infty \exp\left[\mathrm{i}\left(\frac{t^{2k+2}}{2k+2} + \sum_{j=1}^k a_j \frac{t^{2j}}{2j}\right)\right] \mathrm{d}t \\ &= 2\int_0^\infty \exp\left[\mathrm{i}\left(\frac{t^{2k+2}}{2k+2} + \sum_{j=1}^k a_j \frac{t^{2j}}{2j}\right)\right] \mathrm{d}t, \end{aligned} \tag{4}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ is a vector of real numbers.

The "odd" and "even" parts of their names arise because the polynomial phases in the exponential functions of the integrals $O_k(\mathbf{a})$ and $E_k(\mathbf{a})$ are odd and even functions of t, respectively, whereas the "oid" indicates their connection with the unfolding of a cuspoid singularity. When we need to refer to both classes of integrals, we will use the noun *oddenoids* (not to be confused with adenoids).

Although $O_k(\mathbf{a})$ and $E_k(\mathbf{a})$ are special cases of the cuspoid canonical integral (see Section 2.1), it is often convenient to consider them as forming separate classes of canonical integral, because the odd-ness or even-ness of their integrand phases is usually enforced by symmetry in practical applications.

We will be particularly concerned with cases where the orders of $O_k(\mathbf{a})$ and $E_k(\mathbf{a})$ have the values $k = 1, 2$ and 3, since it is very difficult to visualise integrals depending on more than three real parameters.

1.1. Oddenoids of order $k = 1$

The definition (3) shows that the first oddoid integral

$$O_1(a_1) = \int_{-\infty}^\infty \exp\left[\mathrm{i}\left(\frac{t^3}{3} + a_1 t\right)\right] \mathrm{d}t = 2\pi Ai(a_1) \tag{5}$$

is proportional to the regular Airy function, $Ai(a_1)$—also sometimes called the *fold* canonical integral when the terminology of elementary catastrophe theory [42,50] is used for the phase of the integrand. The properties and many applications of the Airy function are well known [5,39,47], and $O_1(a_1)$ is only briefly considered in this paper.