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Type II Hermite–Padé approximation to the exponential function☆

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Dedicated to Nico Temme on the occasion of his 65th birthday.

Abstract

We obtain strong and uniform asymptotics in every domain of the complex plane for the scaled polynomials $a(3nz)$, $b(3nz)$, and $c(3nz)$ where a, b, and c are the type II Hermite–Padé approximants to the exponential function of respective degrees $2n+2$, $2n$ and $2n$, defined by $a(z)\mathrm{e}^{-z} - b(z) = \mathcal{O}(z^{3n+2})$ and $a(z)\mathrm{e}^{z} - c(z) = \mathcal{O}(z^{3n+2})$ as $z \to 0$. Our analysis relies on a characterization of these polynomials in terms of a 3×3 matrix Riemann–Hilbert problem which, as a consequence of the famous Mahler relations, corresponds by a simple transformation to a similar Riemann–Hilbert problem for type I Hermite–Padé approximants. Due to this relation, the study that was performed in previous work, based on the Deift–Zhou steepest descent method for Riemann–Hilbert problems, can be reused to establish our present results.

MSC: 41A21; 41A28; 30E10; 30E25

Keywords: Hermite-Padé approximation; Riemann–Hilbert problems; Steepest descent analysis

1. Hermite–Padé approximation

In this paper we consider quadratic Hermite–Padé approximation to the exponential function. Type I quadratic Hermite–Padé approximation to the exponential function near 0 consists of finding polynomials p_{n_1,n_2,n_3}, q_{n_1,n_2,n_3} and r_{n_1,n_2,n_3} of degrees n_1, n_2 and n_3 respectively, such that

$$p_{n_1,n_2,n_3}(z)\mathrm{e}^{-z} + q_{n_1,n_2,n_3}(z) + r_{n_1,n_2,n_3}(z)\mathrm{e}^{z} = \mathcal{O}(z^{n_1+n_2+n_3+2}), \quad z \to 0.$$

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If we set the right-hand side equal to zero and solve for e^z, then we obtain the algebraic function

$$\frac{-q_{n_1,n_2,n_3}(z) \pm \sqrt{q^2_{n_1,n_2,n_3} - 4p_{n_1,n_2,n_3}(z)r_{n_1,n_2,n_3}(z)}}{2r_{n_1,n_2,n_3}(z)}$$

as an approximation to e^z. Type II Hermite–Padé approximation is simultaneous rational approximation to e^{-z} and e^z and consists of finding polynomials a_{n_1,n_2,n_3}, b_{n_1,n_2,n_3} and c_{n_1,n_2,n_3} of degrees at most n_2+n_3+2, n_1+n_3 and n_1+n_2, respectively, such that

$$\begin{aligned} a_{n_1,n_2,n_3}(z)e^{-z} - b_{n_1,n_2,n_3}(z) &= \mathcal{O}(z^{n_1+n_2+n_3+2}), \quad z \to 0, \\ a_{n_1,n_2,n_3}(z)e^{z} - c_{n_1,n_2,n_3}(z) &= \mathcal{O}(z^{n_1+n_2+n_3+2}), \quad z \to 0. \end{aligned} \tag{1.1}$$

This gives the rational approximants $b_{n_1,n_2,n_3}(z)/a_{n_1,n_2,n_3}(z)$ to e^{-z} and $c_{n_1,n_2,n_3}(z)/a_{n_1,n_2,n_3}(z)$ to e^z, and both rational approximants have the same denominator. It is well-known that for the case of exponentials, all indices n_1, n_2, n_3 are normal, i.e., the polynomials a_{n_1,n_2,n_3}, b_{n_1,n_2,n_3} and c_{n_1,n_2,n_3} exist and are unique up to a normalization constant with degrees exactly n_2+n_3+2, n_1+n_3 and n_1+n_2, see [12, Theorem 2.1, p. 129].

Hermite–Padé approximation to the exponential function have been of interest since Hermite and have recently been investigated in [4,5,8,23,24]. The asymptotic distribution of the zeros for the scaled type I Hermite–Padé polynomials,

$$P_n(z) = p_{n,n,n}(3nz), \quad Q_n(z) = q_{n,n,n}(3nz), \quad R_n(z) = r_{n,n,n}(3nz) \tag{1.2}$$

and their asymptotic behavior as $n \to \infty$, has recently been studied in detail in [17–20,10], see also [9]. In [10] a Riemann–Hilbert problem for type I Hermite–Padé approximation was formulated. The asymptotic analysis of this Riemann–Hilbert problem with the Deift–Zhou [7] steepest descent method for oscillatory Riemann–Hilbert problems and Stahl's geometric description of the problem, allowed the authors of [10] to find strong asymptotic formulas for the polynomials (1.2) as well as for the type I remainder term

$$E_n(z) = P_n(z)e^{-3nz} + Q_n(z) + R_n(z)e^{3nz} \tag{1.3}$$

that hold uniformly in every region of the complex plane. The paper [10] contained the first instance of a steepest descent analysis for a 3×3 matrix-valued Riemann–Hilbert problem. It was followed by [1,3] which dealt with the asymptotic analysis of 3×3 matrix-valued Riemann–Hilbert problems arising in random matrix theory. A Riemann–Hilbert analysis for rational interpolants for the exponential function was carried out in [25].

It is the aim of this paper to show that the Riemann–Hilbert analysis of [10] also produces the corresponding asymptotic results for the scaled type II Hermite–Padé polynomials

$$A_n(z) = a_{n,n,n}(3nz), \quad B_n(z) = b_{n,n,n}(3nz), \quad C_n(z) = c_{n,n,n}(3nz), \tag{1.4}$$

and for the type II remainder terms

$$E_n^{(1)}(z) = A_n(z)e^{-3nz} - B_n(z), \quad E_n^{(2)}(z) = A_n(z)e^{3nz} - C_n(z). \tag{1.5}$$

This is due to the fact that the type II Hermite–Padé polynomials are characterized by a Riemann–Hilbert problem, which is directly related to the Riemann–Hilbert problem for type I Hermite–Padé polynomials. This relation was first observed in [22] and also used in [2,6]. See also Section 2. So we follow the asymptotic analysis of [10], and the reader is advised to consult a copy of that paper too when reading the proofs in this paper.

To illustrate the connection between type I and type II we have depicted in Figs. 1 and 2 the zeros of the type I and type II Hermite–Padé approximants for the case $n_1 = n_2 = n_3 = 60$. As it may be apparent from Figs. 1 and 2, scaled zeros asymptotically accumulate on specific curves in the complex plane, and the same system of curves is relevant for both type I and type II approximants. More precisely, the subsets of zeros on the left and on the right in Figs. 1 and 2, once scaled, tend to the same limit curves as the degree tends to infinity. A similar assertion holds true for some of the zeros of $q_{60,60,60}$ in the middle of Fig. 1 (those outside of the imaginary axis) and the corresponding subsets of zeros of $b_{60,60,60}$ and $c_{60,60,60}$, respectively, lying in the left and right half-planes.