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The role of the Fox–Wright functions in fractional sub-diffusion of distributed order

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Abstract

The fundamental solution of the fractional diffusion equation of distributed order in time (usually adopted for modelling sub-diffusion processes) is obtained based on its Mellin–Barnes integral representation. Such solution is proved to be related via a Laplace-type integral to the Fox–Wright functions. A series expansion is also provided in order to point out the distribution of time-scales related to the distribution of the fractional orders. The results of the time fractional diffusion equation of a single order are also recalled and then re-obtained from the general theory.

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1. Introduction

The Wright function is defined by the series representation, valid in the whole complex plane,

$$W_{\lambda,\mu}(z) := \sum_{k=0}^{\infty} \frac{z^k}{k\Gamma(\lambda k + \mu)}, \quad \lambda > -1, \ \ \mu \in \mathbf{C}, \ \ z \in \mathbf{C}. \tag{1.1}$$

It is an entire function of order $1/(1+\lambda)$, that has been known also as *generalized Bessel function*.[1]

Originally, Wright introduced and investigated this function with the restriction $\lambda \geqslant 0$ in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions [55–57]. Only later, in 1940, he considered the case $-1 < \lambda < 0$ [58]. We note that in the handbook of the Bateman Project see [15, vol. 3, Chapter 18], presumably for a misprint, λ is restricted to be non-negative in spite of the fact that the 1940 Wright's paper is cited.

For the cases $\lambda > 0$ and $-1 < \lambda < 0$ we agree to distinguish the corresponding functions by calling them Wright functions of the first and second type, respectively. As a matter of fact the two types of functions exhibit a quite

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[1] When $\lambda = 1$ the Wright function can be expressed in terms of the Bessel function of order $\nu = \mu - 1$. In fact we have $J_{\mu-1}(z) = (z/2)^{\mu-1} W_{1,\mu}(-z^2/4)$.

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different asymptotic behaviour as it was shown more recently in two relevant papers by Wong and Zaho [53,54]. The case $\lambda = 0$ is trivial since it turns out from (1.1) $W_{0,\mu}(z) = \exp(z)/\Gamma(\mu)$.

Following a former idea of Wright himself [57], the Wright functions can be generalized as follows:

$$ {}_p\Psi_q(z) := \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i k)}{\prod_{j=1}^{q} \Gamma(b_i + B_i k)} \frac{z^k}{k!}, \tag{1.2} $$

where $z \in \mathbf{C}$, $\{a_i, b_j\} \in \mathbf{C}$, $\{A_i, B_j\} \in \mathbf{R}$ with $A_i, B_j \neq 0$ and $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. An empty product, when it occurs, is taken to be 1.

The following alternative notations are commonly used:

$$ {}_p\Psi_q \left[\begin{matrix} (a_i, A_i)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} ; z \right] = {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix} ; z \right]. \tag{1.3} $$

Then, the standard Wright function (1.1), being obtained from (1.2) when $p = 0$ and $q = 1$ with $B_1 = \lambda > -1$, $b_1 = \mu$, reads

$$ W_{\lambda,\mu}(z) \equiv {}_0\Psi_1 \left[\begin{matrix} -- \\ (\mu, \lambda) \end{matrix} ; z \right]. \tag{1.4} $$

All the above functions are known to belong to the more general class of the Fox H functions introduced in 1961 by Fox [16]. For more information the interested reader is referred to the specialized literature including the books [25,28,41,45,50], and the relevant articles [23,24,26,51]. In particular, we recommend the article by Kilbas et al. [26] where the authors have established the conditions for the existence of ${}_p\Psi_q(z)$, see Section 2, and provided its representations in terms of Mellin–Barnes integrals, Section 3, and Fox H functions, Section 4.

For the sake of reader's convenience, we devote the Appendix for a short outline of the H functions in order to understand the Fox representation of the standard and generalized Wright functions of the first and second type that we shall introduce in the following. More appropriately, following [51,13], we can refer to the generalized Wright functions simply to as the Fox–Wright functions.

The Fox notation for the standard Wright functions depends on their type and reads

$$ W_{\lambda,\mu}(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\lambda k + \mu)} = \begin{cases} H^{1,0}_{0,2} \left[-z \left| \begin{matrix} - \quad ; \quad - \\ (0,1); (1-\mu, \lambda) \end{matrix} \right. \right], & \lambda > 0; \\ H^{1,0}_{1,1} \left[-z \left| \begin{matrix} -; (\mu, -\lambda) \\ (0,1); - \end{matrix} \right. \right], & -1 < \lambda < 0. \end{cases} \tag{1.5} $$

Putting $(b_1, B_1) = (\mu, \lambda)$, we have for the generalized Wright function:

$$ {}_p\Psi_q(z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j k)}{\Gamma(\mu + \lambda k) \prod_{j=2}^{q} \Gamma(b_j + B_j k)} \frac{z^k}{k!}, \tag{1.6} $$

$$ {}_p\Psi_q(z) = \begin{cases} H^{1,p}_{p,q+1} \left[-z \left| \begin{matrix} (1-a_j, A_j)_{1,p} \quad ; \quad - \\ (0,1); (1-\mu, \lambda), (1-b_j, B_j)_{2,q} \end{matrix} \right. \right], & \lambda > 0; \\ H^{1,p}_{p+1,q} \left[-z \left| \begin{matrix} (1-a_j, A_j)_{1,p}; (\mu, -\lambda) \\ (0,1); (1-b_j, B_j)_{2,q} \end{matrix} \right. \right], & -1 < \lambda < 0. \end{cases} \tag{1.7} $$

In this paper we shall show the key-role of the standard and generalized Wright functions of the second type for finding the fundamental solutions of diffusion-like equations containing fractional derivatives in time of order $\beta < 1$. In the physical literature, such equations are in general referred to as *fractional sub-diffusion equations*, since they are used as model equations for the kinetic description of anomalous diffusion processes of slow type, characterized by a sub-linear growth of the variance (the mean squared displacement) with time. For an easy introduction to anomalous diffusion and fractional kinetics see the popular articles [29,49].

In addition to the simplest case of a single time-fractional derivative, more generally we can have a weighted (discrete or continuous) spectrum of time-fractional derivatives of distributed order (less than 1): then we speak about *fractional sub-diffusion of distributed order*. We note that only from a few years the fractional diffusion equations of distributed