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ScienceDirect
Journal of Computational and Applied Mathematics 207 (2007) 291 – 300
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Connection factors in the Schrödinger equation with a polynomial potential

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Abstract

Given a second order differential equation with two singular points, namely the origin and infinity, the connection factors allow to split a power series solution into formal solutions with known asymptotic behavior. A procedure is suggested to obtain those factors, as quotients of Wronskians of the mentioned solutions, in the case of a Schrödinger equation with a polynomial potential. Application of the procedure to particular cases, whose connection factors are already known, allows us to obtain new relations for quotients and products of gamma functions.
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MSC: 34M40; 34E05

Keywords: Schrödinger equation; Heun equations; Connection factors; Gamma functions

A considerable number of quantum physical problems consist in solving the Schrödinger differential equation

$$-z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+g(z)w=0, \tag{1}$$

with a polynomial "potential" (including centrifugal and energy terms) that, multiplied by z^2, has the form

$$g(z)=\sum_{s=0}^{2N} g_s z^s, \qquad g_{2N}\neq 0. \tag{2}$$

Among those problems one can find anharmonic oscillators, non-relativistic quark confinement, spherical stark effect, molecular models, etc. Besides, the biconfluent and triconfluent Heun equations [14,8], written in normal form, are particular cases of (1). The condition $g_{2N}\neq 0$ imposed to the potential (2) does not restrict the generality of (1) in view of the possibility of replacing the variable z by its square root.

Solutions of (1) as ascending power series of z can be obtained immediately. Of course, two independent solutions can be found. In the physical problems mentioned above one is interested only in solutions regular at $z=0$, that we denote by w_{reg}. Although such series are convergent for any finite z, they are numerically useful only for small and moderate values of $|z|$. For larger values, asymptotic expansions are more convenient. Olver and Stenger [17] discussed

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doi:10.1016/j.cam.2006.10.008

such kind of expansions for a class of differential equations including that in (1) as a particular case. Two independent solutions w_1 and w_2 exist having asymptotic expansions of the form

$$w_j(z) \sim \exp(\xi_j(z)) \sum_{m=0}^{\infty} a_{m,j} z^{-m}, \quad a_{0,j} \neq 0, \quad j = 1, 2, \tag{3}$$

with the abbreviation

$$\xi_j(z) = \sum_{p=1}^{N} \frac{\alpha_{p,j}}{p} z^p + (\alpha_{0,j} - (N-1)/2) \ln z. \tag{4}$$

By requiring the formal expansion in the right-hand side of (3) to satisfy Eq. (1), we obtain (subscripts j omitted)

$$((z\xi'(z))^2 + z^2\xi''(z) - g(z)) \sum_{m=0}^{\infty} a_m z^{-m} + 2z\xi'(z) \sum_{m=1}^{\infty} (-m) a_m z^{-m} + \sum_{m=1}^{\infty} m(m+1) a_m z^{-m} = 0. \tag{5}$$

The $N+1$ constants α_p appearing in the right-hand side of (4) can be chosen in such a way that the powers $z^{2N}, z^{2N-1}, \ldots, z^N$ in the parenthesis of the first term in (5) disappear. This requirement produces a system of equations

$$\begin{aligned} &(\alpha_N)^2 - g_{2N} = 0, \\ &2\alpha_N \alpha_{N-1} - g_{2N-1} = 0, \\ &2\alpha_N \alpha_{N-2} + (\alpha_{N-1})^2 - g_{2N-2} = 0, \\ &\vdots \\ &2\alpha_N \alpha_0 + \cdots - g_N = 0, \end{aligned} \tag{6}$$

which can be solved successively. There are two sets of solutions, $\{\alpha_{p,1}\}$ and $\{\alpha_{p,2}\}$, that obviously verify

$$\alpha_{p,1} = -\alpha_{p,2}, \quad p = 0, 1, \ldots, N. \tag{7}$$

For each one of those sets of values $\{\alpha_{p,j}\}$, and with an evident notation for the coefficients $\beta_{s,j}$, Eq. (5) can be written in the form

$$\begin{aligned} &\left(\sum_{s=0}^{N-1} \beta_{s,j} z^s\right) \sum_{m=0}^{\infty} a_{m,j} z^{-m} + \left(\sum_{s=1}^{N} 2\alpha_{s,j} z^s + 2\alpha_{0,j} - N + 1\right) \sum_{m=1}^{\infty} (-m) a_{m,j} z^{-m} \\ &\quad + \sum_{m=1}^{\infty} m(m+1) a_{m,j} z^{-m} = 0, \end{aligned} \tag{8}$$

which implies that the coefficients $a_{m,j}$ must satisfy the recurrence relation

$$2\alpha_{N,j} m a_{m,j} = \sum_{s=1}^{N-1} (\beta_{s,j} - 2(m-N+s)\alpha_{s,j}) a_{m-N+s,j} + ((m-N)(m-2\alpha_{0,j}) + \beta_{0,j}) a_{m-N,j}, \tag{9}$$

which allows one, by starting with an arbitrarily chosen $a_{0,j}$, to obtain successively all coefficients $a_{m,j}$.

The solutions to the physical problems mentioned above need to be well behaved (i.e., regular) not only at $z=0$, but also for $z \to \infty$. The behavior for large z of the regular (at $z=0$) solution, w_{reg}, can be immediately determined if one succeeds in writing it as a linear combination of w_1 and w_2,

$$w_{\text{reg}} = T_1 w_1 + T_2 w_2, \tag{10}$$

with coefficients T_1 and T_2 called *connection factors*. The purpose of this work is to present a procedure to calculate these factors. Of course, they can be obtained as quotients of Wronskians,

$$T_1 = \frac{\mathscr{W}[w_{\text{reg}}, w_2]}{\mathscr{W}[w_1, w_2]}, \quad T_2 = \frac{\mathscr{W}[w_{\text{reg}}, w_1]}{\mathscr{W}[w_2, w_1]}. \tag{11}$$

The computation of the denominators in these expressions is immediate from the formal expansions (3). Bearing in mind the relations (4) and (7), an expansion in decreasing powers of z is to be expected. But since the Wronskian

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