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An approximation solution of a nonlinear equation with Riemann–Liouville's fractional derivatives by He's variational iteration method

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Abstract

In this article, an application of He's variational iteration method is proposed to approximate the solution of a nonlinear fractional differential equation with Riemann–Liouville's fractional derivatives. Also, the results are compared with those obtained by Adomian's decomposition method and truncated series method. The results reveal that the method is very effective and simple.

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1. Introduction

Until now in various areas of physics and engineering nonlinear equations, especially fractional differential equation (FDE), were presented. For example the fractional derivative has been accruing in damping laws, motion in Newtonian fluid, dynamical systems, etc. [4,20]. Recently, the numerical solutions of FDE have been established by Diethelm and Ford [5]. Also, the solution of FDE has been obtained through Adomian's decomposition method [3,18,19].

The variational iteration method was first proposed by He [6–9] and was successfully applied to autonomous ordinary differential equation [10], and other fields [12,13].

Recently, new applications of He's variational iteration method were applied to Schrodinger-KdV, generalized KdV and shallow water equations [1], to Burger's and coupled Burger's equations [2], to linear Helmholtz partial differential equation [14] and recently to nonlinear fractional differential equations with Caputo differential derivative [15].

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The mathematical definition of fractional calculus has been the main subject of many different approaches [16,17]. The left-sided Riemann–Lioville fractional integral of order $q>0$ of a function $f(x)$ is defined as [7,17,18]

$$\frac{\mathrm{d}^{-q} f(x)}{\mathrm{d}x^{-q}} = \frac{1}{\Gamma(q)} \int_0^x \frac{f(t)\,\mathrm{d}t}{(x-t)^{1-q}}, \quad x>0,$$

and the Riemann–Liouville's fractional derivative is defined as

$$\frac{\mathrm{d}^{q} f(x)}{\mathrm{d}x^{q}} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\mathrm{d}^{-(n-q)} f(x)}{\mathrm{d}x^{-(n-q)}}\right) = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^x \frac{f(t)\,\mathrm{d}t}{(x-t)^{1-n+q}},$$

where n is an integer that satisfies $n-1 \leqslant q < n$.

2. He's variational iteration method

To illustrate the basic concept of He's variational iteration method, we consider the following nonlinear FDE [7]

$$\mathscr{L}u(t) + \mathscr{N}u(t) = g(t), \tag{1}$$

where $\mathscr{L}$ is a linear operator, $\mathscr{N}$ is a nonlinear operator including fractional differential part, and $g(t)$ is a known analytical function.

Ji-Huan He has modified the general Lagrange multiplier method into an iteration method, which is called correction functional, in the following way [6–10]

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\mathscr{L}u_n(x) + \mathscr{N}\tilde{u}_n(x) - g(x))\,\mathrm{d}x, \tag{2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory [11], the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation [6–10], i.e., $\delta\tilde{u}_n = 0$. It is shown that this method is very effective and easy and can solve a large class of nonlinear problems. For linear problem, its exact solution can be obtained by only one iteration, because λ can be exactly identified.

3. Application

In general, there exists no method that yields an exact solution for nonlinear FDE, only approximate solutions can be obtained. To illustrate the advantages and the accuracy of He's variational iteration method, we will consider the following nonlinear FDE which was recently solved by Adomian's decomposition method (ADM) and compared by truncated series method in [18].

Example (*Saha Ray and Bera [18]*). Consider the nonlinear FDE

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\mathrm{d}^{1/2}u}{\mathrm{d}t^{1/2}} - 2u^2 = 0. \tag{3}$$

First, we obtain the solution by truncated series method. We can look at the solution $u(t)$ of Eq. (3) in the form of the fractional power series

$$u(t) = \sum_{i=0}^{\infty} y_i t^{i/2}, \tag{4}$$