Available online at www.sciencedirect.com
ScienceDirect
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
Journal of Computational and Applied Mathematics 203 (2007) 32–56
www.elsevier.com/locate/cam

New conservative schemes with discrete variational derivatives for nonlinear wave equations☆

Takayasu Matsuo

Interfaculty Initiative in Information Studies, The University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan

Received 29 November 2005; received in revised form 13 March 2006

Abstract

New conservative finite difference schemes for certain classes of nonlinear wave equations are proposed. The key tool there is "discrete variational derivative", by which discrete conservation property is realized. A similar approach for the target equations was recently proposed by Furihata, but in this paper a different approach is explored, where the target equations are first transformed to the equivalent system representations which are more natural forms to see conservation properties. Applications for the nonlinear Klein–Gordon equation and the so-called "good" Boussinesq equation are presented. Numerical examples reveal the good performance of the new schemes.

MSC: 65M06

Keywords: Finite-difference method; Conservation; Nonlinear wave equation; Boussinesq equation

1. Introduction

The numerical integration of the one-dimensional nonlinear wave equations of the form

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\delta G}{\delta u}, \quad 0 < x < L, \quad t > 0 \tag{P1}$$

and

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\frac{\delta G}{\delta u}, \quad 0 < x < L, \quad t > 0 \tag{P2}$$

is considered, where $G(u, u_x)$ is a real-valued function of $u(x, t)$ and $u_x = \partial u/\partial x$, and

$$\frac{\delta G}{\delta u} = \frac{\partial G}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial G}{\partial u_x}$$

☆ This work is supported by the 21st Century COE Program on Information Science and Technology Strategic Core, and by the Grant-in-Aid for Scientific Research (A), (B), (S), and for Encouragement of Young Scientists (B) of the Japan Society for the Promotion of Science.
E-mail address: matsuo@mist.i.u-tokyo.ac.jp.

0377-0427/$ - see front matter
doi:10.1016/j.cam.2006.03.009

is the variational derivative of $G(u, u_x)$. The nonlinear Klein–Gordon equation, for example, belongs to (P1), and some class of the Boussinesq equations belongs to (P2). The equations of the form (P1) have in common the "energy" conservation property:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left(\frac{1}{2}(u_t)^2 + G(u, u_x)\right) \mathrm{d}x = 0, \tag{1}$$

under some suitable boundary conditions, and thus called "conservative". Eqs. (P2) are also conservative, but their conservation properties are not as simple as (1). This will be discussed in the next section.

For such conservative equations, it is preferable that numerical schemes have discrete analogues of the conservation properties, since they often yield physically correct results and also numerical stability [2]. Such schemes are called "conservative schemes". In early phase of these researches, many attempts to find conservative schemes were done independently for several specific problems; for example, conservative schemes for the nonlinear Klein–Gordon equation were studied in [1,4,11,20] (see also references in [7,17]). In the end of the 20th century, a more unified method was given in [7,8,17], by which conservative schemes for wide range of problems can be constructed automatically. Most of specific conservative schemes in the literature then turned out to be examples of the unified approach. The method targets conservative (or dissipative, where the "energy" is monotonically dissipated along the solution) partial differential equations which is defined with variational derivative; Furihata [7] targeted real-valued equations of the form

$$\frac{\partial u}{\partial t} = (-1)^{s+1}\left(\frac{\partial}{\partial x}\right)^s \frac{\delta G}{\delta u}, \quad s = 0, 1, 2, \ldots,$$

which is conservative when s is odd, and dissipative otherwise; Matsuo and Furihata [17] targeted complex-valued equations of the form

$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\frac{\delta G}{\delta \overline{u}} \quad \text{and} \quad \frac{\partial u}{\partial t} = -\frac{\delta G}{\delta \overline{u}},$$

which is conservative and dissipative, respectively ($\delta G/\delta\overline{u}$ is complex variational derivative); Furihata then targeted Eqs. (P1) in [8]. In these studies the key concept is "discrete variational derivative", which is discrete analogue of variational derivative. The numerical scheme is then defined with it analogously to the original equation so that the discrete conservation property should be "inherited".

There are three aims in this paper. The first aim is to introduce a new approach for Eqs. (P1), which is different from Furihata [8]. The key there is the fact that Eqs. (P1) can be represented as systems of first-order differential equations, by appropriately introducing intermediate variables. Discretizing these systems using the idea of discrete variational derivative not only gives rise to new families of conservative scheme, but brings an additional advantage that in some new schemes the time mesh size can be adaptively changed. The second aim is to cover Eqs. (P2) which was not covered in Furihata [8]. In particular, conservative schemes for the Boussinesq equations are obtained for the first time in the literature as far as the author knows. The third, somewhat subsidiary aim is to clarify the relation between Furihata's approach [8] (we call it the "previous" approach throughout this paper), and the new approach. Both approaches utilizes the idea of discrete variational derivative, but start with different representations of the target equations. Then arises a natural question: do the resulting schemes by the different approaches coincide just as the continuous equations do? The "staggered grid" technique is introduced to discuss this issue.

This paper is organized as follows; in Section 2 the target equations and their properties are reviewed; Section 3 is devoted to the summary of the discrete symbols and Furihata's previous approach; then in Section 4 the new schemes are presented and the relation between the previous and new approaches is discussed; Section 5 is for application examples where, in particular, conservative schemes for the Boussinesq equations are given; Section 6 is for concluding remarks.

2. Target equations

In this section the target equations and their properties are summarized.