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Optimal shape of a strongest inverted column

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Abstract

By using Pontryagin's maximum principle we determine the shape of the strongest column positioned in a constant gravity field, simply supported at the lower end and clamped at upper end (with the possibility of axial sliding). It is shown that the cross-sectional area function is determined from the solution of a nonlinear boundary value problem. A variational principle for this boundary value problem is formulated and two first integrals are constructed. These integrals lead to an a priori estimate of the value of one the missing initial condition and to the reduction of the order of the system. The optimal shape of a column is determined by numerical integration.

MSC: 49J15; 74K10

Keywords: Optimal shape; Pontryagin's principle; First integrals

1. Introduction

The problem that we shall treat in this note may be considered as a version of tallest column problem. Recall the tallest column problem was formulated in [6]. The tallest column is a homogeneous column made of a given volume (mass) of material (for example, equal to unity) and being so shaped that it does not buckle under its own weight although it is higher than any other column made of the same volume of material. Such column is called the optimally shaped column. Keller and Niordson [6] determined that the height of the optimally shaped column is 2.034 times larger than the height of the column having constant cross-section and being made of the same amount of material as the optimally shaped column. After the work of Keller and Niordson the tallest column problem has not been subject of further analysis, until the works of McCarthy [7,8] and Cox and McCarthy [4]. As a matter of fact, Cox and McCarthy [4] state that "the tallest column problem appears to have started and ended with the work of Keller and Niordson". McCarthy [7,8] made progress in solving the problem since the shape of the tallest column was determined, numerically, for several values of parameters. We note that the tallest column problem is equivalent to the lightest column problem (a column having given height and being stable against buckling in a constant gravity field while any other column of given weight would buckle under the same conditions).

Our intention in this note is to solve the problem of the *lightest* column in a constant gravity field for the case when the boundary conditions used in [6,7] are interchanged, i.e., upper end is fixed (with the possibility of axial sliding)

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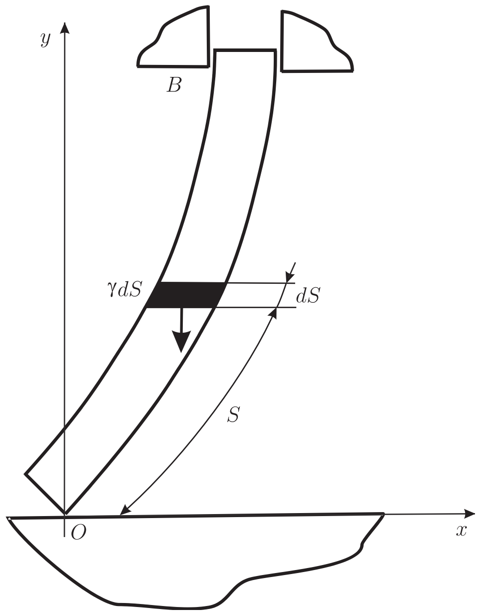

Fig. 1. Coordinate system and load configuration.

lower end is simply supported with possibility of horizontal sliding (see Fig. 1). Such boundary conditions and the stability bound for the case of a column with constant cross-section were treated in [15,14] (see also [1]). We shall use Pontryagin's maximum principle [11] to formulate the optimality condition. This condition will be the same as the one presented in [8]. However, our further analysis will differ from the one presented in [7,8]. Namely we use the variational structure of the relevant equations to construct a first integral of Jacobi type (see [12]) and by using this integral with the suitable combination of differential equations, we shall obtain another first integral. With these integrals we will be able to determine a priori estimate of one "missing" initial condition in the problem. Also by use of the first integrals we will be eliminating one variable and checking the accuracy of numerical integration by calculating integrals at each step of integration and comparing the value with the known constant.

We shall assume that buckling load is an isolated eigenvalue of the equilibrium equations. This assumption was also used in [6]. The delicate analysis of Cox and McCarthy presented in [4] (see also [9,10] for similar problem) did not lead to the definite answer whether or not the column in [6] buckles at an eigenvalue. Use of the existence arguments similar to those presented in [4] may lead to a conclusion about spectrum in our case. However, such analysis is beyond the scope of this paper.

2. Formulation

Consider a column of height L shown in Fig. 1. Suppose that the column is in uniform gravity field with acceleration g. The equilibrium equations read [1]

$$\frac{\mathrm{d}H}{\mathrm{d}S} = q_0 A, \quad \frac{\mathrm{d}V}{\mathrm{d}S} = 0, \quad \frac{\mathrm{d}M}{\mathrm{d}S} = -V\cos\vartheta + H\sin\vartheta, \tag{1}$$

where $q_0 = \rho g$ with ρ being the mass density, H and V are components of the contact force $\mathbf{F}$ along the $\bar{x}$ and $\bar{y}$ axes, respectively, S is the arc-length measured from the point O, M is the bending moment, ϑ is the angle between the tangent to the column axis and $\bar{x}$ axis and S is the arc-length of the rod axis measured from the origin of the rectangular Cartesian coordinate system $\bar{x} - O - \bar{y}$. Note that H is the weight of the column above section defined by S. For the case of classical Bernoulli–Euler rod, we adjoin to (1) the geometrical

$$\frac{\mathrm{d}x}{\mathrm{d}S} = \cos\vartheta, \quad \frac{\mathrm{d}y}{\mathrm{d}S} = \sin\vartheta, \tag{2}$$