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Convergence of the variants of the Chebyshev–Halley iteration family under the Hölder condition of the first derivative☆

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Abstract

The present paper is concerned with the convergence problem of the variants of the Chebyshev–Halley iteration family with parameters for solving nonlinear operator equations in Banach spaces. Under the assumption that the first derivative of the operator satisfies the Hölder condition of order p, a convergence criterion of order $1+p$ for the iteration family is established. An application to a nonlinear Hammerstein integral equation of the second kind is provided.

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1. Introduction

Let X and Y be (real or complex) Banach spaces, $\Omega \subseteq X$ be an open subset and let $F : \Omega \subseteq X \to Y$ be a nonlinear operator on Ω. Finding solutions of the nonlinear operator equation

$$F(x) = 0 \tag{1.1}$$

in Banach spaces is a very general subject which is widely used in both theoretical and applied areas of mathematics. When F is Fréchet differentiable, the most important method to find the approximation solution is Newton's method. One of the famous results on Newton's method is the well-known Kantorovich theorem (cf. [21]) which guarantees convergence of Newton's sequence to a solution under very mild conditions. Further researches on Newton's method are referred to [24–26].

As it is well known, in the case when F has the second continuous Fréchet derivative on Ω, there are several kinds of cubic generalizations for Newton's method. The most important two are the Chebyshev method and the Halley method, see e.g., [1–4,17,20,28], respectively. Another more general family of the cubic extensions is the family of Chebyshev–Halley-type methods, which was proposed in [13] by Gutiérrez and Hernández. This family includes the

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Chebyshev method ($\alpha = 0$) and the Halley method ($\alpha = \frac{1}{2}$) as well as the convex acceleration of Newton's method (or the super-Halley method) ($\alpha = 1$, cf. [5,15,14]) as its special cases and has been explored extensively in [13,16,27]. Let $\alpha \in [0, 1]$. Then the family of Chebyshev–Halley-type methods is defined by

$$x_{\alpha,n+1} = x_{\alpha,n} - \left[\mathbf{I} + \tfrac{1}{2}L_F(x_{\alpha,n})[\mathbf{I} - \alpha L_F(x_{\alpha,n})]^{-1}\right] F'(x_{\alpha,n})^{-1}F(x_{\alpha,n}), \quad n = 0, 1, \ldots, \tag{1.2}$$

where $\mathbf{I}$ is the identity and, for each $x \in X$, $L_F(x)$ is a bounded linear operator from X to Y defined by

$$L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x), \quad x \in X. \tag{1.3}$$

Recent interests are focused on the study of the variants of the Chebyshev iteration and the Halley iteration as well as the convex acceleration of Newton's method, which are obtained by replacing the second derivative in (1.3) with the difference of the first derivatives at x and z:

$$F''(x)(z - x) \approx F'(z) - F'(x),$$

where $z = x + \lambda(-F'(x)^{-1}F(x))$ while $\lambda \in (0, 1]$ is a parameter. This is equivalent that $L_F(x)$ is replaced by the bounded linear operator $H(x, y) : X \to Y$ defined by

$$H(x, y) = \frac{1}{\lambda}F'(x)^{-1}[F'(x + \lambda(y - x)) - F'(x)], \tag{1.4}$$

where $y = x - F'(x)^{-1}F(x)$.

Such a variant has the advantage that avoids the computation of the second derivatives (so works for operators with the first derivatives only) but keeps the higher orders of convergence. The variant of the convex acceleration of Newton's method was first presented by Ezquerro and Hernández in [7], where a cubical convergence criterion based on the Lipschitz constant and the boundary of the second derivative was established under the assumption that the second derivative of F satisfies the Lipschitz condition. The same variant was done in [18] for the Chebyshev method, and cubical convergence criterions for this variant were studied in [18,19]. Convergence criterions under the Lipschitz condition of the first derivative were discussed for the variants of the convex acceleration of Newton's method, the Chebyshev method and the Halley method, respectively, in [8,18,30].

The variant of the family of Chebyshev–Halley-type methods was presented in [29]. Under the assumption that the second derivative F'' satisfies the Hölder condition on some suitable closed ball $\mathbf{B}(x_0, R)$:

$$\|F''(x) - F''(y)\| \leqslant K\|x - y\|^p \quad \text{for all } x, y \in \mathbf{B}(x_0, R), \tag{1.5}$$

a unified convergence criterion depending on the values of the operator, its first derivative and second derivative at the initial point x_0 as well as the Hölder constant K was established for the variant.

The present paper is a continuation of the paper [29]. More precisely, just assuming that the first derivative F' satisfies the Hölder condition on some suitable closed ball $\mathbf{B}(x_0, R)$:

$$\|F'(x) - F'(y)\| \leqslant K\|x - y\|^p \quad \text{for all } x, y \in \mathbf{B}(x_0, R) \tag{1.6}$$

(its second derivative is not necessary), we establish a unified convergence criterion only depending on the values of the operator and its first derivative at the initial point x_0 as well as the Hölder constant for the variant of the family of Chebyshev–Halley-type methods. The main theorem is stated in Section 3, which includes the corresponding results for the variant of the convex acceleration of Newton's method and the variant of the Chebyshev method as well as the variant of the Halley method obtained in [8,18,30] as special examples. An application to a nonlinear Hammerstein integral equation of the second kind (cf. [22]) is given in the final section.

We should compare the convergence criterion in the present paper with that in [29]. The main difference between them is that we use condition (1.6) here instead of (1.5). Clearly, (1.5) implies (1.6) (with different constant K). In particular, in the case when F'' does not exist or F'' is unbounded, condition (1.5) is not satisfied. Section 4 of the present paper provides such an example (cf. Example 4.1), where the operator F has the first derivative F' satisfying (1.6) (so the convergence criterion in the present paper may be applicable) but does not have the second derivative on any closed ball containing x_0 and the convergence criterion in [29] is not applicable.