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Scaled generalized Bernstein polynomials and containment regions for the zeros of polynomials

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Abstract

Let $P_n(x) = \sum_{k=0}^{n} \beta_k b_{k,n}(x; \alpha, \beta)$, $\beta_n \neq 0$, where $b_{k,n}(x; \alpha, \beta) := (\alpha + x)^k (\beta - x)^{n-k}$, $k = 0, 1, \ldots, n$, $\alpha, \beta \in \mathbb{C}$, $\alpha \neq -\beta$, a generalized scaled Bernstein polynomial. Extending a result of Winkler [A resultant matrix for scaled Bernstein polynomials, Linear Algebra Appl. 319 (2000) 179–191] a companion matrix for P_n is given. The application of some matrix methods to this companion matrix yields regions for the zeros of P_n.

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1. Introduction

Let

$$P_n(x) := \sum_{k=0}^{n} \beta_k b_{k,n}(x; \alpha, \beta) := \sum_{k=0}^{n} a_k x^k, \quad \widehat{P}_n(x) := \sum_{k=0}^{n} \beta_k x^k, \tag{1}$$

where

$$b_{k,n}(x; \alpha, \beta) := (\alpha + x)^k (\beta - x)^{n-k}, \quad k = 0, 1, \ldots, n, \quad \alpha, \beta \in \mathbb{C}, \quad \alpha \neq -\beta, \quad \beta_n = 1,$$

be a generalized scaled Bernstein polynomial which also is given in its power (monomial) basis expansion of degree n. The classical scaled Bernstein polynomials we get for $\alpha = 0$, $\beta = 1$.

There are many contributions which concern containment regions for zeros of polynomials, which are given in their power basis expansion (cf. [7–19]). Rather short is the list of contributions which concerns containment regions for zeros of polynomials, which are given in expansions in other basis functions (cf. [1,20–22]). In this paper we consider generalized scaled Bernstein polynomials of type (1). In Section 2 for convenience the connection between the coefficients of a generalized scaled Bernstein polynomial and the coefficients in power basis expansion is given.

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In Section 3 first we observe that with $\phi(x) := (\beta x - \alpha)/(1+x)$ for an arbitrary companion matrix A of $\widehat{P_n}$ then $\phi(A)$ is a companion matrix of P_n, and for an inclusion region D for the zeros of $\widehat{P_n}$ then $\phi(D)$ is an inclusion region for the zeros of P_n. The most common case of an annulus is considered in Proposition 1. Second, we consider a special companion matrix A of $\widehat{P_n}$ which results from a similarity transformation of the usual Frobenius companion matrix and thus get in $\phi(A)$ a companion matrix for the scaled generalized Bernstein polynomials extending a result of Winkler [23]. A representation of this companion matrix as a sum of a diagonal matrix, a nilpotent matrix, and a matrix of rank 1 makes it possible to compute the traces of $\phi(A)$, $\phi(A)^2$, and $\phi(A)^*\phi(A)$ in terms of the coefficients of the generalized scaled Bernstein polynomial. With this from results of Wolkowicz and Styan [25] we get bounds for the zeros of P_n. Furthermore, in Section 4 with the aid of the numerical range we give a circle in the complex plane which contains the numerical range of the companion matrix $\phi(A)$ and thus the zeros of the given scaled generalized Bernstein polynomial P_n.

2. Scaled generalized Bernstein polynomials

Let $\alpha, \beta \in \mathbb{C}$, $\alpha \neq -\beta$. For $n \in \mathbb{N}$, we consider the generalized basis Bernstein functions

$$b_{k,n}(x;\alpha,\beta) := (\alpha + x)^k(\beta - x)^{n-k}, \quad k = 0, 1, \ldots, n. \tag{2}$$

The polynomials $b_{k,n}(.;\alpha,\beta)$, $k = 0, 1, \ldots, n$, are linearly independent—that is, they are a basis of the vector space of all polynomials of degree at most n.

Let

$$P_n(x) := \sum_{k=0}^{n} a_k x^k \tag{3}$$

be a polynomial of degree n. Because of the basis property of the generalized Bernstein functions P_n can be expressed according to

$$P_n(x) := \sum_{k=0}^{n} \beta_k b_{k,n}(x;\alpha,\beta). \tag{4}$$

Then we have

$$\beta_k = \sum_{v=0}^{k} \sum_{\mu=v}^{n} \binom{n-v}{k-v} a_\mu \binom{\mu}{v} (\alpha+\beta)^{v-n}(-1)^{\mu-v}\alpha^{\mu-v}, \quad k = 0, 1, \ldots, n,$$

and

$$a_k = \sum_{v=0}^{n} \beta_v \sum_{\mu=0}^{k} (-1)^\mu \binom{v}{k-\mu}\binom{n-v}{\mu} \alpha^{v-k+\mu}\beta^{n-v-\mu}, \quad k = 0, 1, \ldots, n.$$

If $\alpha = 0$, then

$$\beta_k = \sum_{v=n-k}^{n} \binom{v}{n-k} a_{n-v}\beta^{-v}, \quad k = 0, 1, \ldots, n,$$

and

$$a_k = \sum_{v=0}^{n} \beta_v (-1)^{k-v} \binom{n-v}{k-v} \beta^{n-k}, \quad k = 0, 1, \ldots, n,$$

and if in addition $\beta = 1$, then

$$\beta_k = \sum_{v=n-k}^{n} \binom{v}{n-k} a_{n-v}, \quad k = 0, 1, \ldots, n,$$