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R-K type Landweber method for nonlinear ill-posed problems

L. Li*, B. Han, W. Wang

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

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Abstract

In this paper we propose the R-K type Landweber iteration and investigate the convergence of the method for nonlinear ill-posed problem $F(x) = y$ where $F : H \to H$ is a nonlinear operator between Hilbert space H. Moreover, for perturbed data with noise level δ we prove that the convergence rate is $O(\delta^{2/3})$ under appropriate conditions. Finally, the numerical performance of this R-K type Landweber iteration for a nonlinear convolution equation is compared with the Landweber iteration.

MSC: 32H50; 34K29

Keywords: Ill-posed problems; R-K type Landweber method; Continuous methods

1. Introduction

Let us consider a nonlinear operator equation

$$F(x) = y, \quad F : H \to H \tag{1.1}$$

in a real Hilbert space H (Eq. (1.1) in a complex Hilbert space can be treated similarly), where F is a nonlinear operator with domain H with corresponding inner products $(\cdot, \cdot)$ and norms $\|\cdot\|$, respectively. Throughout this paper we assume that $y^\delta \in H$ are the available approximate data with

$$\|y - y^\delta\| \leqslant \delta, \tag{1.2}$$

where δ denotes the noise level, that (1.1) has a solution x^* (which need not be unique) and F possesses a locally uniformly bounded *Fréchet*-derivative $F'(\cdot)$ in a ball $B_r(x_0)$ of radius r around $x_0 \in H$.

In the theory of ill-posed problems many methods for nonlinear ill-posed problems are known. One of the best understood regularization theory for nonlinear ill-posed inverse problems is the method of Tikhonov regularization [5,2]. In contrast to Tikhonov regularization, iteration methods [6,4] produce an approximation to the solution within every iteration step. Several iteration methods for nonlinear operators were under investigation during the last years. In the paper of Hanke et al. [4] the well known Landweber iteration for linear ill-posed problems [3] has been extended to the nonlinear case.

* Corresponding author.
E-mail addresses: lily93480@sohu.com (L. Li), bohan@hit.edu.cn (B. Han).

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There have some achievements to the study of the dynamical system up to now. Airapetyan and Ramm [1] have posed a general approach to continuous analogs of discrete methods and established fairly general convergence theorems aiming at the following dynamical system:

$$\dot{x}(t) = \Phi(x(t), t), \quad x(0) = x_0, \tag{1.3}$$

where Φ is a nonlinear operator, $\Phi : H \times [0, +\infty) \to H$. Meanwhile, they constructed the discrete schemes generated by this continuous process:

$$x_{k+1} = x_k + \omega\Phi(x_k, t_k), \quad k = 0, 1, 2, \ldots \;. \tag{1.4}$$

In 2003, Ramm [7] proved the global convergence for ill-posed equations with monotone operators.

In [9], Tautenhahn studied the continuous Landweber method:

$$\dot{x}(t) = -F'(x(t))^*[F(x(t)) - y], \quad x(0) = x_0 \in H. \tag{1.5}$$

Here $x_0 \in H$ is some element. (In this case the noise level $\delta = 0$.)

When the noise level $\delta \neq 0$, a regularized approximation $x^\delta(T)$ of x^* is obtained by solving the initial value problem:

$$x^\delta(t) := F'(x^\delta(t))^*[y^\delta - F(x^\delta(t))], \quad 0 < t \leqslant T, \ x^\delta(0) = x_0, \tag{1.6}$$

where T plays the role of the regularization parameter. If we use Euler's method with a step size $r = 1$ to discrete (1.6), we can obtain the usual Landweber iteration:

$$x_{k+1}^\delta = x_k^\delta - F'(x_k^\delta)^*[F(x_k^\delta) - y^\delta].$$

Tautenhahn uses the following assumption to study the convergence of continuous Landweber method:

(A1) In a Ball $B_r(x_0)$ of radius r around x_0 there holds:

$$\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\| \leqslant \eta \|F(x) - F(\tilde{x})\|, \quad \eta < 1$$

for all $x, \tilde{x} \in B_r(x_0) \subset H$. This assumption guarantees that for all $x, \tilde{x} \in B_r(x_0)$ there holds

$$\frac{1}{1+\eta}\|F'(x)(x - \tilde{x})\| \leqslant \|F(x) - F(\tilde{x})\| \leqslant \frac{1}{1-\eta}\|F'(x)(x - \tilde{x})\|. \tag{1.7}$$

Proposition 1 in [9] shows that the discrepancy $\|F(x^\delta(T)) - y^\delta\|$ as a function of T is monotone non-increasing. Furthermore, it shows that the error $\|x^\delta(T) - x^*\|$ as a function of T is strong monotonically decreasing as far as $\|F(x^\delta(T)) - y^\delta\| \geqslant \tau\delta$ holds with $\tau = (1 + \eta)/(1 - \eta)$. Hence, it makes sense to choose the regularization parameter in (1.5) from a discrepancy principle, i.e., $T = T^*$ is a solution of the nonlinear equation

$$h(T) = \|F(x^\delta(T)) - y^\delta\| - \tau\delta = 0, \tag{1.8}$$

with $\tau > (1 + \eta)/(1 - \eta)$.

We also know that under some conditions Eq. (1.8) has a unique solution $T^* < \infty$ from Proposition 2 in [2].

In the following two theorems Tautenhahn proved convergence properties of method (1.6) when noise level $\delta = 0$ and $\delta \neq 0$, respectively.

Theorem 1.1 (*Tautenhahn [9]*). *Let* (1.2) *and* (A1) *be satisfied. If* (1.1) *is solvable in* $B_r(x_0)$, *then*

$$x(T) \to x^* \quad \textit{for } T \to \infty \tag{1.9}$$

(*convergence for exact data*), *where* $x^* \in B_r(x_0)$ *is a solution of* (1.1). *Let* $x^\dagger$ *denote the unique solution of minimal distance to* x_0, *then, if in addition* $N(F'(x^\dagger)) \subset N(F'(x))$ *for all* $x \in B_r(x_0)$, *then* $x(T)$ *converges to* $x^\dagger$.

In this paper, $N(\cdot)$ denotes the null space of an operator.