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Analysis and finite element approximation of a Ladyzhenskaya model for viscous flow in streamfunction form☆

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Abstract

In this paper we consider a model for the motion of incompressible viscous flows proposed by Ladyzhenskaya. The Ladyzhenskaya model is written in terms of the velocity and pressure while the studied model is written in terms of the streamfunction only. We derived the streamfunction equation of the Ladyzhenskaya model and present a weak formulation and show that this formulation is equivalent to the velocity–pressure formulation. We also present some existence and uniqueness results for the model. Finite element approximation procedures are presented. The discrete problem is proposed to be well posed and stable. Some error estimates are derived. We consider the 2D driven cavity flow problem and provide graphs which illustrate differences between the approximation procedure presented here and the approximation for the streamfunction form of the Navier–Stokes equations. Streamfunction contours are also displayed showing the main features of the flow.

MSC: 65N30; 76M10; 78M10; 76D03; 35Q35

Keywords: Finite element method; Ladyzhenskaya model; Streamfunction formulation

1. Introduction

In [23–25], Ladyzhenskaya has proposed a model for the motion of ideal incompressible flow. An excellent piece of motivation why one consider this model can be found in [9]. Du and Gunzburger mentioned several reasons to consider this model. They are modeling, mathematical, practical engineering and practical programming point of views. Ladyzhenskaya presented her model in velocity–pressure version. Further studies are made in [8–10,26].

In this paper, we study the streamfunction equation of Ladyzhenskaya model. The attractions of the streamfunction equation are that the incompressibility constraint is automatically satisfied, the pressure is not present in the weak form and there is only one scalar unknown to solve for. The purpose of this paper is to present and analyze a weak formulation for the streamfunction of the Ladyzhenskaya model and its discretization.

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We first need to state the Ladyzhenskaya model in velocity–pressure form. Let Ω be a bounded, simply connected, polygonal domain in R^2 and $\vec{u}$ denotes the velocity field, p the pressure and $\vec{f}$ the body force. The Ladyzhenskaya equations for 2D incompressible fluid flow are

$$-\partial_x(\hat{A}(u)u_{1,x}) - \partial_y(\hat{A}(u)u_{1,y}) + u_1u_{1,x} + u_2u_{1,y} + p_x = f \quad \text{in } \Omega, \tag{1}$$

$$-\partial_x(\hat{A}(u)u_{2,x}) - \partial_y(\hat{A}(u)u_{2,y}) + u_1u_{2,x} + u_2u_{2,y} + p_y = f \quad \text{in } \Omega, \tag{2}$$

$$u_x + u_y = 0 \quad \text{on } \partial\Omega, \tag{3}$$

with the homogeneous Dirichlet boundary conditions on u_1 and u_2, i.e.,

$$u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \tag{4}$$

where in (2)

$$\hat{A}(\vec{u}) = \varepsilon_0 + \varepsilon_1|\bigtriangledown\vec{u}|^{q-2} \quad \text{with } q > 2, \tag{5}$$

and

$$|\bigtriangledown\vec{u}| = [u_{1,x}^2 + u_{1,y}^2 + u_{2,x}^2 + u_{2,y}^2]^{1/2}. \tag{6}$$

We also assume that $1/Re = \varepsilon_0 > 0$ and $\varepsilon_1 > 0$ are constants. Note that if we set $\varepsilon_1 = 0$, Eqs. (1)–(3) become the familiar Navier–Stokes equations.

Any divergence-free velocity field, $\vec{u}$, in $H_0^1(\Omega)$ has a streamfunction ψ defined by

$$\vec{\text{curl}}\,\psi = \vec{u}.$$

Moreover, ψ is uniquely determined up to a constant. Since $\partial\psi/\partial\tau = 0$ on $\partial\Omega$, where τ denotes the unit tangent to $\partial\Omega$, setting $\psi = 0$ on $\partial\Omega$ guarantees the uniqueness of the streamfunction.

Thus we have

$$-\partial_x(A(\psi)\psi_{xy}) - \partial_y(A(\psi)\psi_{yy}) + \psi_y\psi_{xy} - \psi_x\psi_{yy} + p_x = f_1 \quad \text{in } \Omega, \tag{7}$$

$$-\partial_x(A(\psi)\psi_{xx}) - \partial_y(A(\psi)\psi_{xy}) + \psi_y\psi_{xx} - \psi_x\psi_{xy} + p_y = f_2 \quad \text{in } \Omega, \tag{8}$$

$$\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{9}$$

where in (7) and (8), n represents the outward unit normal to Ω and $A(\psi)$ is defined by

$$A(\psi) = \varepsilon_0 + \varepsilon_1\|\vec{\triangle}\psi\|^{q-2}, \tag{10}$$

and

$$\vec{\triangle}\psi = \vec{\text{grad}}\;(\vec{\text{grad}}\,\psi) = [\psi_{xx}, \psi_{xy}, \psi_{yx}, \psi_{yy}]^{\mathrm{T}},$$

and

$$\|\vec{\triangle}\psi\| = [\psi_{xx}^2 + 2\psi_{xy}^2 + \psi_{yy}^2]^{1/2}. \tag{11}$$

Taking the "curl" of (7) and (8) will eliminate the pressure p and yields the streamfunction equation of the Ladyzhenskaya equations

$$\begin{gathered}\partial_{xx}(A(\psi)\psi_{xx}) + 2\partial_{xy}(A(\psi)\psi_{xy}) + \partial_{yy}(A(\psi)\psi_{yy}) - \psi_y\triangle\psi_x + \psi_x\triangle\psi_y = f_{2,x} - f_{1,y} \quad \text{in } \Omega,\\ \psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{12}$$