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Lattice Boltzmann model for two-dimensional unsteady Burgers' equation ☆

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Abstract

In this paper, a special lattice Boltzmann model is proposed to simulate two-dimensional unsteady Burgers' equation. The maximum principle and the stability are proved. The model has been verified by several test examples. Excellent agreement is obtained between numerical predictions and exact solutions. The cases of steep oblique shock waves are solved and compared with the two-point compact scheme results. The study indicates that lattice Boltzmann model is highly stable and efficient even for the problems with severe gradients.

MSC: 74S30; 65M12; 34E05

Keywords: Lattice Boltzmann method; Two-dimensional Burgers' equation; 4-bit model; Multi-scale analysis; Maximum principle

1. Introduction

In recent years, the lattice Boltzmann method (LBM) has been developed into an alternative and promising numerical scheme for simulating fluid flows [1,8,3] and solving various mathematical–physical problems [7,4,10–12]. This method can be either regarded as an extension of the lattice gas automaton [6] or as a special discrete form of the Boltzmann equation for kinetic theory [5]. Unlike conventional numerical schemes based on discretizations of partial differential equations describing macroscopic conservation laws, the LBM is based on solving the discrete-velocity Boltzmann equation from statistical physics. It describes the microscopic picture of particles movement in an extremely simplified way, while on the macroscopic level it gives a correct average description.

The Burgers' equation, which is also called the nonlinear advection–diffusion equation, is a simplified model of Navier–Stokes equations. It retains the nonlinear aspects of the governing equation in many practical transport problems such as aggregation interface growth, the formation of large-scale structures in the adhesion model for cosmology, turbulence transport, shock wave theory, wave processes in thermoelastic medium, transport and dispersion of pollutants

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in rivers and sediment transport. Therefore, it is usually used to test different numerical methods. The unsteady two-dimensional Burgers' equation in one unknown variable take the following form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=v\frac{\partial^2 u}{\partial x^2}+v\frac{\partial^2 u}{\partial y^2},\quad x_0\leqslant x\leqslant x_N,\quad y_0\leqslant y\leqslant y_N,\quad t>0, \tag{1}$$

with the initial condition $u(x,y,0)=u_0(x,y)$. Here the viscous coefficient $v=1/Re>0$, Re is the Reynolds number. For a small value of v, Burgers' equation behaves merely as hyperbolic partial differential equation and the problem becomes very difficult to solve as a steep shock-like wave fronts developed.

Elton [4] and Shen et al. [11] have proposed lattice Boltzmann models for 2D Burgers' equation in which there is only one convective term. In the paper, we developed a 4-bit model for Eq. (1). By using Taylor expansion and multi-scale analysis, the time-dependent two-dimensional Burgers' equation is recovered from the lattice Boltzmann equation, and the local equilibrium distribution functions are obtained. It is generally recognized that the LBM is a Lagrangian discretization of a discrete-velocity Boltzmann equation. In this view, we find such lattice Boltzmann scheme satisfies maximum principle, therefore, we complete the proof of stability.

The rest of the paper is organized as follows. Section 2 proposes a lattice Boltzmann model and derives the 2D Burgers' equation from the model. A stability analysis of the LBM is also given in Section 3. In Section 4 some numerical experiments are made using our model. And the conclusions are given in the end.

2. Lattice Boltzmann method

According to the theory of the LBM, it consists of two steps: (1) streaming, where each particle moves to the nearest node in the direction of its velocity; and (2) colliding, which occurs when particles arriving at a node interact and possibly charge their velocity directions according to scattering rules. Usually, with the single-relaxation-time BGK approximation [2], these two steps can be combined into the following LBE:

$$f_\alpha(\mathbf{x}+\mathbf{e}_\alpha\Delta t,t+\Delta t)-f_\alpha(\mathbf{x},t)=-\frac{1}{\tau}(f_\alpha-f_\alpha^{\mathrm{eq}}), \tag{2}$$

where f_α is the distribution function of particles; f_α^{eq} is the local equilibrium distribution function of particles; Δx and Δt are space and time increments, respectively; $c=\Delta x/\Delta t$ is "the speed of light" in the system; $\mathbf{e}_\alpha$ is the velocity vector of a particle in the α link and τ is the dimensionless single-relaxation-time which controls the rate of approach to equilibrium. Virtually it is a full discretization of time, space, and velocity. The macroscopic velocity, u is defined in terms of the distribution functions as

$$u=\sum_\alpha f_\alpha=\sum_\alpha f_\alpha^{\mathrm{eq}}=\sum_\alpha f_\alpha^{(0)}. \tag{3}$$

The lattice Boltzmann schemes are established on the square grids with four perpendicular directions:

$$e_1=(c,0),\quad e_2=(0,c),\quad e_3=(-c,0),\quad e_4=(0,-c).$$

This is a 4-bit model shown in Fig. 1. To derive the macroscopic equation from the lattice BGK model, we employ the Taylor expansion and multi-scale analysis. The distribution functions are expanded up to linear terms in the small expansion parameter ε

$$f_\alpha=f_\alpha^{(0)}+\varepsilon f_\alpha^{(1)}+\mathrm{O}(\varepsilon^2).$$

From the kinetic equation (2), we expand the distribution function $f_\alpha(x+\Delta t e_\alpha,t+\Delta t)$ in its Taylor expansion and calculate an approximation of $f_\alpha^{(1)}$,

$$\begin{aligned}f_\alpha(\mathbf{x}+\mathbf{e}_\alpha\Delta t,t+\Delta t)&=f_\alpha(\mathbf{x},t)+\Delta t e_{\alpha i}\partial_{x_i}f_\alpha+\Delta t\partial_t f_\alpha+\mathrm{O}(\varepsilon^2)\\&=\left(1-\frac{1}{\tau}\right)f_\alpha(\mathbf{x},t)+\frac{1}{\tau}f_\alpha^{\mathrm{eq}}(\mathbf{x},t)\end{aligned}$$

$$\varepsilon f_\alpha^{(1)}=-\tau\Delta t(e_{\alpha i}\partial_{x_i}f_\alpha+\partial_t f_\alpha)+\mathrm{O}(\varepsilon^2), \tag{4}$$