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# Error estimators for advection–reaction–diffusion equations based on the solution of local problems

Rodolfo Araya[a,1], Edwin Behrens[b,*,2], Rodolfo Rodríguez[a,3]

[a] $GI^2MA$, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
[b] Facultad de Ingeniería, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile

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**Abstract**

This paper deals with a posteriori error estimates for advection–reaction–diffusion equations. In particular, error estimators based on the solution of local problems are derived for a stabilized finite element method. These estimators are proved to be equivalent to the error, with equivalence constants eventually depending on the physical parameters. Numerical experiments illustrating the performance of this approach are reported.


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## 1. Introduction

This paper deals with the advection–diffusion–reaction equations. This kind of problems arise in many applications, for instance, to model pollutant transport and degradation in aquatic media, which was the motivation of the present work.

In applications, typically the advective or reactive terms are dominant. In this case, inner or boundary layers arise and stabilization techniques have to be used to avoid spurious oscillations (see [4] and references therein, which include numerical methods). When finite element methods are used, adequately refined meshes are useful to improve the quality of the numerical solution with minimal computational effort. These schemes are typically based on a posteriori error indicators. We refer to [3] for a survey of this kind of techniques applied to advection dominated problems. See also [5,6,9] for error estimates in alternative norms adequately suited to this kind of equations.

In a recent article [1], we have introduced and analyzed from theoretical and experimental points of view an adaptive scheme to efficiently solve the advection–reaction–diffusion equation. This scheme is based on a stabilized finite element method introduced in [2] combined with a residual error estimator, similar to another one introduced in [8].

* Corresponding author.
*E-mail address:* ebehrens@ucsc.cl (E. Behrens).
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We have proved global upper and local lower error estimates in the energy norm, with constants which depend on the shape-regularity of the mesh, the polynomial degree of the finite element approximating space, and the local mesh Peclet number.

Following this line, we introduce in this paper a framework to derive error estimators based on the solution of local problems. We prove the equivalence of the resulting estimators with the residual based estimator analyzed in [1] and, hence, with the energy norm of the error.

We report several numerical experiments which allow us to assess the effectiveness of this approach to capture boundary and inner layers very sharply and without significant oscillations. The experiments also show that the schemes lead to optimal orders of convergence.

The paper is organized as follows. In Section 2 we recall the advection–diffusion–reaction problem under consideration and the stabilized scheme. In Section 3 we recall the main result of [1] and derive a posteriori error estimators based on the solution of local problems. Then, we prove their equivalence with the residual error estimator analyzed in [1]. Finally, in Section 4, we report the results of some numerical tests, to assess the performance of the estimators.

## 2. A stabilized method for a model problem

Our model problem is the advection–reaction–diffusion problem

$$\begin{cases} -\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u + bu = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \varepsilon\dfrac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$, is a bounded polygonal domain with a Lipschitz boundary $\Gamma = \bar{\Gamma}_{\mathrm{D}} \cup \bar{\Gamma}_{\mathrm{N}}$, with $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$. We denote by $\boldsymbol{n}$ the outer unit normal vector to $\Gamma$.

We are interested in the advection–reaction dominated case and assume that:

(A1) $\varepsilon \in \mathbb{R}$: $0 < \varepsilon \ll 1$;
(A2) $\boldsymbol{a} \in \mathrm{W}^{1,\infty}(\Omega)^2$: $\operatorname{div}\boldsymbol{a} = 0$ in $\Omega$;
(A3) $b \geqslant 1$ in $\Omega$;
(A4) $\Gamma_{\mathrm{D}} \supset \{\boldsymbol{x} \in \Gamma\text{: } \boldsymbol{a}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) < 0\}$;
(A5) $f \in \mathrm{L}^2(\Omega)$, $g \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$.

We use standard notation for Sobolev and Lebesgue spaces, norms, and inner products. Moreover, we introduce the following notation: Let

$$\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega) := \{\varphi \in \mathrm{H}^1(\Omega)\text{: } \varphi = 0 \text{ on } \Gamma_{\mathrm{D}}\}$$

and $B$ be the bilinear form defined on $\mathrm{H}^1(\Omega)$ by

$$B(v, w) := \int_\Omega (\varepsilon\nabla v\cdot\nabla w + \boldsymbol{a}\cdot\nabla v\, w + bvw).$$

Then, the standard variational formulation of problem (2.1) is the following: Find $u \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that

$$B(u, v) = \int_\Omega f v + \int_{\Gamma_{\mathrm{N}}} g v \quad \forall v \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega). \tag{2.2}$$

We consider the following (energy) norm on $\mathrm{H}^1(\Omega)$:

$$|||u||| := (\varepsilon\|\nabla u\|^2_{0,\Omega} + \|u\|^2_{0,\Omega})^{1/2}.$$

Assumptions (A1)–(A4) and integration by parts imply that

$$B(v, v) \geqslant |||v|||^2 \quad \forall v \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$$