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Point estimation of simultaneous methods for solving polynomial equations: A survey (II)

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Dedicated to Professor Stephen Smale on the occasion of his 75th birthday

Abstract

The construction of computationally verifiable initial conditions which provide both the guaranteed and fast convergence of the numerical root-finding algorithm is one of the most important problems in solving nonlinear equations. Smale's "point estimation theory" from 1981 was a great advance in this topic; it treats convergence conditions and the domain of convergence in solving an equation $f(z)=0$ using only the information of f at the initial point z_0. The study of a general problem of the construction of initial conditions of practical interest providing guaranteed convergence is very difficult, even in the case of algebraic polynomials. In the light of Smale's point estimation theory, an efficient approach based on some results concerning localization of polynomial zeros and convergent sequences is applied in this paper to iterative methods for the simultaneous determination of simple zeros of polynomials. We state new, improved initial conditions which provide the guaranteed convergence of frequently used simultaneous methods for solving algebraic equations: Ehrlich–Aberth's method, Ehrlich–Aberth's method with Newton's correction, Börsch-Supan's method with Weierstrass' correction and Halley-like (or Wang–Zheng) method. The introduced concept offers not only a clear insight into the convergence analysis of sequences generated by the considered methods, but also explicitly gives their order of convergence. The stated initial conditions are of significant practical importance since they are computationally verifiable; they depend only on the coefficients of a given polynomial, its degree n and initial approximations to polynomial zeros.

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Keywords: Point estimation theory; Localization of zeros; Polynomial zeros; Simultaneous methods; Guaranteed convergence; Initial conditions

1. Point estimation theory based on sequence approach

One of the most important problems in solving nonlinear equations is stating such initial conditions which provide the guaranteed convergence of the applied numerical algorithm. Evidently, only those conditions which depend on attainable data are useful from a practical point of view. First results which deal with computationally verifiable initial conditions providing the guaranteed convergence were stated and developed in [12,31–35]. The research on

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this topic was later continued in [3,7,8,11,13,17–30,37,38,36], and other papers. This approach, often referred to as "point estimation theory", considers convergence conditions and the domain of convergence in solving an equation $f(z)=0$ using only the information of f at the initial point z_0. In this way, it overcomes difficulties that appear in the traditional treating the convergence conditions based on the asymptotical convergence analysis. This analysis is only of theoretical importance since it involves (in the estimation procedure) some unknown parameters as constants, even the (unknown) roots of equation, or uses the terminology as "sufficiently good (close enough) approximations" without quantitative (and computationally verifiable) characterization of the closeness of these approximations to the roots. A review of Smale's point estimation theory and related results can be found in [22,23, Chapters 1–3] and we omit details in this paper.

The study of a general problem of the construction of initial conditions and the choice of initial approximations furnishing guaranteed convergence of a root-finding method is very difficult, even in the case of algebraic polynomials. In this particular case, these conditions should depend only on the coefficients of a given polynomial $P(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ of degree n and the vector of initial approximations $\mathbf{z}^{(0)}=(z_1^{(0)},\ldots,z_n^{(0)})$. More details about the point estimation theory and its applications to algebraic polynomials can be found in the aforementioned papers and the references cited therein.

The paper [22] gives a survey of results concerning the guaranteed convergence of some frequently used iterative methods for the simultaneous determination of polynomial zeros as Durand–Kerner's method, Börsch-Supan's method, the square-root one parameter family. That study uses the concept of convergent iterative corrections proposed in [18]. In this paper, which can be regarded as the continuation of research presented in [22], we state another approach to the convergence analysis in the light of Smale's point estimation theory and based on convergent sequences and some results concerned with the localization of polynomial zeros. Using this approach we improve computationally verifiable initial conditions for several iterative methods which belong to the class of the most efficient and often used simultaneous methods for finding polynomial zeros. The introduced concept presents not only a clear insight into the convergence analysis of sequences produced by the considered methods, but also explicitly gives their order of convergence, which is the advantage in reference to the approach exposed in [22] and some other papers. It is worth noting that the aim of this paper is not only the demonstration of the point estimation theory applied to algebraic polynomials, but also the significant improvement of initial conditions for the four frequently used simultaneous methods for finding polynomial zeros.

The essential question in stating initial convergence conditions is how to express these conditions. The requested form should be computationally verifiable and, in addition, it must take into account some important properties as distribution of zeros, their separation and closeness to initial approximations. Let $I_n:=\{1,\ldots,n\}$ be the index set. For $i\in I_n$ and $m=0,1,\ldots$ let us introduce the quantity

$$W_i^{(m)}=\frac{P(z_i^{(m)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i^{(m)}-z_j^{(m)})}\quad (i\in I_n,\ m=0,1,\ldots)$$

which is often called Weierstrass' correction since it appeared in Weierstrass' paper [40]. As shown in [18], the above requirements can be fulfilled in a satisfactory way by expressing initial conditions in the form

$$w^{(0)}\leqslant c_n d^{(0)}, \tag{1.1}$$

where

$$w^{(0)}=\max_{1\leqslant i\leqslant n}|W_i^{(0)}|,\quad d^{(0)}=\min_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}}|z_i^{(0)}-z_j^{(0)}|$$

and c_n is a real quantity depending only on the polynomial degree n. The use of form (1.1) is justified since it involves the requested properties; indeed, if the initial approximations are close enough, then the minimal distance $d^{(0)}$ can be regarded as a measure of separation of the zeros, while $w^{(0)}$ is related to the closeness of approximations to the zeros.

In [36] Wang and Zhao improved Smale's result for Newton's method and applied it to the Durand–Kerner method for the simultaneous determination of polynomial zeros. Their approach also led in a natural way to form (1.1). In both cases the quantity c_n is expressed as $c_n=1/(an+b)$, where a and b are suitably chosen positive constants.