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Asymptotic approximations for the first incomplete elliptic integral near logarithmic singularity

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Abstract

We find two convergent series expansions for Legendre's first incomplete elliptic integral $F(\lambda, k)$ in terms of recursively computed elementary functions. Both expansions are valid at every point of the unit square $0 < \lambda, k < 1$. Truncated expansions yield asymptotic approximations for $F(\lambda, k)$ as λ and/or k tend to unity, including the case when logarithmic singularity $\lambda = k = 1$ is approached from any direction. Explicit error bounds are given at every order of approximation. For the reader's convenience we present explicit expressions for low-order approximations and numerical examples to illustrate their accuracy. Our derivation is based on rearrangements of some known double series expansions, hypergeometric summation algorithms and inequalities for hypergeometric functions.

MSC: 33E05; 33C75; 33F05

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1. Introduction

Legendre's incomplete elliptic integral (EI) of the first kind is defined by [1, (12.2.7)]

$$F(\lambda, k) = \int_0^\lambda \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{1}$$

It is one of the three canonical forms given by Legendre in terms of which all EIs can be expressed. We will only consider the most important case $0 \leqslant k \leqslant 1$, $0 \leqslant \lambda \leqslant 1$.

The subject of series expansions and asymptotic approximations for the first incomplete EI has a long history. An expansion given by Kaplan in 1948 [10] implicitly contained an asymptotic approximation for $F(\lambda, k)$ near the singular point $\lambda = k = 1$ (see (4) below). Soon after Kaplan's paper various series expansions for the first incomplete EI

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were given by Radon in [20] and Kelisky in [11]. A bit later Carlson showed in [3] that $F(\lambda, k)$ can be expressed in terms of Appell's first hypergeometric series F_1 (see [7]), which automatically led to several series expansions through known transformation formulas for F_1. In the same paper he noted that one can derive rapidly convergent expansions by first expressing Legendre's incomplete EIs in a different form. This form had later become known as symmetric standard EIs. Carlson proved that instead of three Legendre's EIs one can use three symmetric standard EIs as canonical forms. The first symmetric standard EI is defined by [3–6]

$$R_F(x, y, z) = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t+x)(t+y)(t+z)}}. \tag{2}$$

It is symmetric in x, y and z, homogenous in all variables of degree $-\frac{1}{2}$ and related to $F(\lambda, k)$ by

$$F(\lambda, k) = \lambda R_F(1-\lambda^2, 1-k^2\lambda^2, 1). \tag{3}$$

Asymptotic formulas for $F(\lambda, k)$ near the point $(1, 1)$ appeared in [2,4,8,17], but the first complete asymptotic series with error bounds at each order of approximation was given by Carlson and Gustafson in terms of the first symmetric standard EI R_F in [5]. As is clear from (3) and homogeneity, the case $\lambda, k \to 1$ for $F(\lambda, k)$ is equivalent to the case $z \to \infty$ with bounded x and y for $R_F(x, y, z)$. The first two approximations from [5] translated into our notation read

$$F(\lambda, k) = \lambda \ln \frac{4}{\sqrt{1-\lambda^2}+\sqrt{1-k^2\lambda^2}} + \theta_1 F(\lambda, k), \tag{4}$$

with relative error bound

$$\frac{(2-\lambda^2(1+k^2))\ln(1-k^2\lambda^2)}{4\ln[(1-k^2\lambda^2)/16]} < \theta_1 < \frac{2-\lambda^2(1+k^2)}{4}, \tag{5}$$

and

$$F(\lambda, k) = \frac{\lambda}{4}\Bigg[(6-\lambda^2(1+k^2))\ln\frac{4}{\sqrt{1-\lambda^2}+\sqrt{1-k^2\lambda^2}} - 2 + \lambda^2(1+k^2)$$
$$+\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}\Bigg] + \theta_2 F(\lambda, k), \tag{6}$$

with relative error bound

$$\frac{9(1-k^2\lambda^2)^2\ln(1-k^2\lambda^2)}{64\ln[(1-k^2\lambda^2)/16]} < \theta_2 < \frac{3(1-k^2\lambda^2)^2}{8}. \tag{7}$$

The authors also provide more precise approximations at the price of having the first complete EI in them

$$F(\lambda, k) = \frac{2}{\pi}K\left(\sqrt{1-k^2}\right)\ln\frac{4}{\sqrt{1-\lambda^2}+\sqrt{1-k^2\lambda^2}} - \delta_1 \tag{8}$$

$$= \frac{2}{\pi}K\left(\sqrt{1-k^2}\right)\ln\frac{4}{\sqrt{1-\lambda^2}+\sqrt{1-k^2\lambda^2}} - \frac{1}{4}\left(2-\lambda^2-k^2\lambda^2-\sqrt{(1-\lambda^2)(1-k^2)}\right) + \delta_2, \tag{9}$$

where absolute errors have bounds given by

$$\frac{1-k^2\lambda^2}{8} < \delta_1 < \frac{(1-k^2\lambda^2)\ln(4)}{k^2\lambda^2}, \quad \frac{9(1-k^2\lambda^2)^2}{64} < \delta_2 < \frac{3(1-k^2\lambda^2)^2\ln(2)}{2k^2\lambda^2}. \tag{10}$$