Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 202 (2007) 352–376
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

On the asymptotic integration of nonlinear differential equations

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Received 13 December 2004; received in revised form 21 November 2005

Abstract

The aim of the present paper is twofold. Firstly, the paper surveys the literature concerning a specific topic in asymptotic integration theory of ordinary differential equations: the class of second order equations with Bihari-like nonlinearity. Secondly, some general existence results are established with regard to a condition that has been found recently to be of significant use in the theory of elliptic partial differential equations.

Keywords: Nonlinear differential equations; Fixed point theory; Asymptotic integration

1. History and further developments

In 1941, Caligo [16] published a paper in which it was established that, given a continuous real-valued function $A(t)$ defined in the positive half axis, the solutions of the linear equation

$$y''(t) + A(t)y(t) = 0, \quad t > 0, \tag{1}$$

satisfy the condition

$$\lim_{t\to+\infty} y'(t) = \lim_{t\to+\infty} \frac{y(t)}{t} \in \mathbb{R} \tag{2}$$

provided that

$$|A(t)| < \frac{l}{t^{2+\rho}} \quad \text{for all large } t, \tag{3}$$

where $l, \rho > 0$ are given. Furthermore, if $\rho > 1$, the solutions can be represented asymptotically as

$$y(t) = c_1 t + c_2 + \mathrm{o}(1) \quad \text{when } t \to +\infty, \quad c_{1,2} \in \mathbb{R}. \tag{4}$$

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0377-0427/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.cam.2005.11.038

The case when $\rho \in (0, 1]$ has been answered in the negative with regard to the development (4) via an example. The results of Caligo extended previous theorems by Dini, Kneser, Sansone commented in his paper.

The proofs in [16] rely on establishing the solution of a system of integral equations by means of an iterative process [16, p. 290] as well as on a special estimate [16, p. 294] of the type

$$\frac{|y(t)|}{t} < m + \frac{1}{2} \sup_{s\in[t_0,T]} \left[\frac{|y(s)|}{s}\right] \quad \text{for } t \in [t_0, T], \quad m > 0. \tag{5}$$

The case of (3) when $\rho > 1$ was also discussed by Bitterlich–Willmann [12] in 1941 with similar conclusions, namely the existence of

$$\lim_{t\to+\infty} y'(t), \qquad \lim_{t\to+\infty} [y(t) - ty'(t)].$$

A thorough investigation and extension of these results has been done by Haupt [40].

Caligo's work raised a lot of interest. In their paper from 1942, Boas et al. [13] extended the conclusions of [16] about (2) to the larger class of linear equations

$$y''(t) + A(t)y(t) = B(t), \quad t > 0, \tag{6}$$

where $A(t)$ and $B(t)$ are continuous and real-valued and

$$\int_0^{+\infty} t|A(t)|\,\mathrm{d}t < +\infty, \qquad \int_0^{+\infty} B(t)\,\mathrm{d}t \text{ exists,}$$

by means of estimate (5) (see [13, Lemma, p. 847]). Introducing the functions $A_{1,2}(t)$,

$$A_1(t) = \tfrac{1}{2}[|A(t)| + A(t)] = \max[A(t), 0],$$
$$A_2(t) = \tfrac{1}{2}[|A(t)| - A(t)] = -\min[A(t), 0],$$

the conclusion (2) holds for the solutions of (6) even if only ([13, p. 849])

$$\limsup_{t\to+\infty} t \int_t^{+\infty} A_1(s)\,\mathrm{d}s < 1, \qquad \int_0^{+\infty} tA_2(t)\,\mathrm{d}t < +\infty.$$

Moreover, if

$$\int_0^{+\infty} tA_1(t)\,\mathrm{d}t = +\infty$$

then $\lim_{t\to+\infty} y'(t) = 0$ for every solution $y(t)$ of (6). Also, following [86, p. 366], when

$$\int_0^{+\infty} A_2(t)\,\mathrm{d}t < +\infty,$$

all the solutions $y(t)$ of (1) satisfy

$$y'(t) = \mathrm{O}(t^{-1/2}) \quad \text{as } t \to +\infty$$

(this reads as $A(t)$ *belongs to the class* $\frac{1}{2}$ in Wintner's terminology) if and only if $A_1(t)$ is in the same class. For more details about the preceding conditions on $A(x)$ in the case of (1), see the 1955 paper of Hartman and Wintner [36].

The results of Caligo, Boas, Boas Jr., Levinson were extended to nth order inhomogeneous linear differential equations by Wilkins Jr. [85] and Haupt [41]. Other significant results in this direction are due to Ghizzetti [29] and Hartman [38] (see also their references pointing out papers by Dunkel and Faedo).

In 1947, Bellman published a memoir [8] on the asymptotic behavior of solutions of linear differential equations where Haupt's result [41, p. 290] was addressed by means of *integral inequalities*. Precisely, the proof of [8, Theorem 7] relies on an application of the well-known Gronwall–Bellman integral inequality [32,7] to get a global estimate of

$$|y'(t)| \quad \text{and} \quad \frac{|y(t)|}{t}, \quad t > 0,$$

for the solutions $y(t)$ of (1) (see [8, Eq. (8.11)]).