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Unified treatment for the evaluation of generalized complete and incomplete gamma functions

I.I. Guseinov[a,*], B.A. Mamedov[b]

[a]Department of Physics, Faculty of Arts and Sciences, Onsekiz Mart University, Çanakkale, Turkey
[b]Department of Physics, Faculty of Arts and Sciences, Gaziosmanpasa University, Tokat, Turkey

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Abstract

The present work applies the binomial expansion theorems to evaluate the generalized complete and incomplete gamma functions arising in the wave scattering and diffraction theory. A simple and efficient algorithm for the calculation of these functions is developed. Some numerical results are presented for significant mapping examples and they are briefly discussed. The formulas obtained are numerically stable for all values of parameters occurring in generalized complete and incomplete gamma functions.

Keywords: Gamma function; Exponential integral function; Binomial expansion theorem; Scattering and diffraction

1. Introduction

The generalized gamma functions are widely used in the solution of many problems of wave scattering and diffraction theory [11,12]. To date, a few various methods have been developed for the analysis of the generalized gamma functions [1,10–14]. Some fundamental properties of these functions were investigated in [11]. In this paper, a new approach to the computation of the generalized complete and incomplete gamma functions are proposed, which considerably improved its capabilities during numerical evaluations in significant cases. Using binomial expansion theorem the generalized gamma and incomplete gamma functions are expressed through the familiar gamma and exponential integral functions whose analytical procedures can be found in our recently published papers [5,6].

2. Definitions and recurrence relations for generalized gamma functions

The generalized complete and incomplete gamma functions examined in this work are defined in [11]

$$\Gamma_\beta(\alpha, c) = \int_0^\infty \frac{t^{\alpha-1}\mathrm{e}^{-t}}{(t+c)^\beta}\,\mathrm{d}t, \tag{1a}$$

* Corresponding author. Tel.: +5335 57 8593.
E-mail address: iguseinov@cheerful.com (I.I. Guseinov).

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$$\gamma_\beta(\alpha, x, c) = \int_0^x \frac{t^{\alpha-1} e^{-t}}{(t+c)^\beta} dt, \tag{2a}$$

$$\Gamma_\beta(\alpha, x, c) = \int_x^\infty \frac{t^{\alpha-1} e^{-t}}{(t+c)^\beta} dt, \tag{3a}$$

where $\alpha = n + \varepsilon \geqslant 1$ $(n = 1, 2, \ldots, 0 \leqslant \varepsilon < 1)$, $\beta = m + \delta \geqslant 0$ $(m = 1, 2, \ldots, -1 \leqslant \delta \leqslant 0)$ and $c > 0$. These functions are related to each other by

$$\Gamma_\beta(\alpha, c) = \gamma_\beta(\alpha, x, c) + \Gamma_\beta(\alpha, x, c). \tag{4a}$$

The familiar $\Gamma(\alpha)$, $\gamma(\alpha, x)$ and $\Gamma(\alpha, x)$ functions defined in [2]

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \tag{1b}$$

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt, \tag{2b}$$

$$\Gamma(\alpha, x) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \tag{3b}$$

$$\Gamma(\alpha) = \gamma(\alpha, x) + \Gamma(\alpha, x), \tag{4b}$$

are the special cases of generalized gamma functions for $\beta = 0$. These functions, Eqs. (1b)–(4b), have already been investigated by numerous authors with different algorithms (see, e.g., Ref. [9] and references therein). In our paper [5], for direct calculation of the complete and incomplete gamma functions $\Gamma(\alpha)$, $\gamma(\alpha, x)$ and $\Gamma(\alpha, x)$, the upward and downward recursion, and analytical relations were derived. These relations are used in the evaluation of generalized gamma functions defined by Eqs. (1a)–(3a).

The generalized gamma functions satisfy the following recursive relations:

$$\Gamma_\beta(\alpha, c) = \frac{\delta_{\alpha 1}}{(\beta-1)c^{\beta-1}} - \frac{1}{\beta-1}\Gamma_{\beta-1}(\alpha, c) + \frac{\alpha-1}{\beta-1}\Gamma_{\beta-1}(\alpha-1, c), \tag{5}$$

$$\gamma_\beta(\alpha, x, c) = \frac{\delta_{\alpha 1}}{(\beta-1)c^{\beta-1}} - \frac{x^{\alpha-1}e^{-x}}{(\beta-1)(x+c)^{\beta-1}} - \frac{1}{\beta-1}\gamma_{\beta-1}(\alpha, x, c) + \frac{\alpha-1}{\beta-1}\gamma_{\beta-1}(\alpha-1, x, c), \tag{6}$$

$$\Gamma_\beta(\alpha, x, c) = \frac{x^{\alpha-1}e^{-x}}{(\beta-1)(x+c)^{\beta-1}} - \frac{1}{\beta-1}\Gamma_{\beta-1}(\alpha, x, c) + \frac{\alpha-1}{\beta-1}\Gamma_{\beta-1}(\alpha-1, x, c), \tag{7}$$

where

$$\delta_{\alpha 1} = \begin{cases} 1 & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha \neq 1 \end{cases}.$$

These formulae can easily be established from Eqs. (1a)–(3a) integration by parts using $u = t^{\alpha-1}e^t$ and $dv = (t+c)^{-\beta} dt$.

3. Binomial series expansion relations

In order to express the integrals (1a)–(3a) in terms of complete and incomplete gamma functions we shall use the following well known binomial expansion theorem (see [4,3,8,7]):

$$(x \pm y)^n = \sum_{m=0}^\infty (\pm 1)^m F_m(n) x^{n-m} y^m, \tag{8}$$