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Approximations to -, di- and tri-logarithms

Wadim Zudilin

Department of Mechanics and Mathematics, Moscow Lomonosov State University, Vorobiovy Gory, GSP-2, 119992 Moscow, Russia

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Abstract

We propose hypergeometric constructions of simultaneous approximations to polylogarithms. These approximations suit for computing the values of polylogarithms and satisfy 4-term Apéry-like (polynomial) recursions.

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The series for the logarithm function

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1,$$

is so simple and nice that mathematicians immediately generalize it by introducing the polylogarithms

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad |z| < 1, \quad s = 1, 2, \ldots, \tag{1}$$

and considering then multiple, q-basic, p-adic and any other possible generalizations, just to have a serious mathematical research (i.e., to have some fun). It is so that nobody could now overview the whole amount of the results around all these generalizations of the logarithm, since the literature on the subject increases to infinity as a geometric progression (almost hypergeometrically).

It is not surprising that the transcendence number theory also dreams of getting new and new results for the values of the polylogarithms, especially after Lindemann's proof of the transcendence of $\log x$ for any algebraic x different from 0 and 1. The main problems (or, if you like, intrigues) are therefore extensions of the result to the values of (1), where Lindemann's argument based on proving the transcendence for the inverse, exponential, function does not work in an

E-mail address: wadim@ips.ras.ru
URL: http://wain.mi.ras.ru/.

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obvious manner any more. And we now have so far irrationality and linear independence results for the polylogarithm values at non-zero rational points close to zero, thanks to contributions of Maier [10], Galochkin [6], Nikishin [12], Chudnovsky [5], Hata [7,8], Rhin and Viola [15]. The very last piece of news is the irrationality of $\mathrm{Li}_2(1/q)$ for q integer, $q \leqslant -5$ or $q \geqslant 6$, obtained by the powerful arithmetic method in [15], which improves the range of [8] by adding $q = 6$ (the work [15] also includes quantitative improvements of the irrationality in other cases, but we do not touch this subject in this short note). Another direction of arithmetic investigations are the values of (1) at $z = 1$ (or $z = -1$), so-called *zeta values*. This goes back to Euler's time, who has contributed by the formula

$$\mathrm{Li}_{2k}(1) = \zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2(2k)!} \quad \text{for } k = 1, 2, \ldots, \tag{2}$$

where $B_{2k} \in \mathbb{Q}$ are the Bernoulli numbers, thus Lindemann's proof of the transcendence of π results in the transcendence of the numbers (2). Apéry [1] has shown that $\zeta(3)$ is irrational, and since that time, thanks to Ball and Rivoal [3], we dispose of only partial irrationality information for other values of $\mathrm{Li}_{2k+1}(1) = \zeta(2k + 1)$ if $k = 2, 3, 4, \ldots$.

All known achievements in this subject are closely related to hypergeometric series and also multiple and complex integrals originated from the series. This is a general concept of the hypergeometric method developed for arithmetic study of the values of the polylogarithms; we refer the reader to a brief exposition of this concept in [11].

Here, we would like to present some new ingredients of the hypergeometric method. We cannot achieve some new number-theoretic results by these means, and for the moment this note may be viewed as a methodological contribution. Nevertheless, approximations to the values of the polylogarithms that we derive here are quite reasonable from the computational point of view, and, in this sense, we continue our previous work on deducing curious Apéry-like recurrences.

We hope that the reader is somehow familiar with our work on the hypergeometric method in arithmetic study of zeta values (at least with the preprint [16]).

1. Simultaneous approximations to the logarithm and dilogarithm

For each $n = 0, 1, \ldots,$ consider the rational function

$$R_n(t) = \frac{((t-1)(t-2)\cdots(t-n))^2}{n! \cdot t(t+1)\cdots(t+n)}.$$

Since degree of the numerator is greater than degree of the denominator, we will have a polynomial part while decomposing into partial fractions. The arithmetic properties of this decomposition are given in the following statement; D_n denotes the least common multiple of the numbers $1, 2, \ldots, n$.

Lemma 1. *We have*

$$R_n(t) = \frac{((t-1)(t-2)\cdots(t-n))^2}{n! \cdot t(t+1)\cdots(t+n)} = \sum_{k=0}^{n} \frac{A_k}{t+k} + B(t),$$

where numbers A_k are all integers and $D_n \cdot B(t)$ is an integer-valued polynomial of degree $n - 1$.

Proof. Write this decomposition as follows:

$$R_n(t) = \sum_{k=0}^{n} \frac{A_k}{t+k} + \sum_{j=0}^{n-1} B_j \frac{t(t+1)\cdots(t+j-1)}{j!}$$

(the empty product for $j = 0$ is 1). The coefficients A_k are easily determined by the standard procedure:

$$A_k = R_n(t)(t+k)|_{t=-k} = (-1)^k \binom{n}{k} \binom{n+k}{k}^2, \quad k = 0, 1, \ldots, n, \tag{3}$$