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Multivariate Frobenius–Padé approximants: Properties and algorithms☆

Ana C. Matos

Laboratoire Paul Painlevé UMR CNRS 8524 UFR de Mathématiques Pures et Appliquées, Université des Sciences et Technologies de Lille, France

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Abstract

The aim of this paper is to construct rational approximants for multivariate functions given by their expansion in an orthogonal polynomial system. This will be done by generalizing the concept of multivariate Padé approximation. After defining the *multivariate Frobenius–Padé approximants*, we will be interested in the two following problems: the first one is to develop recursive algorithms for the computation of the value of a sequence of approximants at a given point. The second one is to compute the coefficients of the numerator and denominator of the approximants by solving a linear system. For some particular cases we will obtain a displacement rank structure for the matrix of the system we have to solve. The case of a Tchebyshev expansion is considered in more detail.

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1. Introduction

Let us consider a two variable function $f(x, y)$ given by its expansion (or the first coefficients of its expansion) in an orthogonal polynomial system $\{P_k\}$

$$f(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} P_i(x) P_j(y).$$

We want to construct rational approximants for f by generalizing the concept and ideas of Padé approximation—rational approximation for power series (see for instance [1,5]).

The univariate case has been studied in [21]. There, the *Padé–Legendre approximants* for a series $f(x)=\sum_{i=0}^{\infty} c_i P_i(x)$ have been defined and different types of algorithms for their recursive computation have been proposed. For some classes of functions, acceleration results have been obtained, that is, it has been shown that the sequence of approximants converges to f faster than the partial sums. The numerical examples were very good and incited us to generalize these ideas to the case of a vector function. The *simultaneous Frobenius–Padé approximants* were defined in [22] where their

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E-mail address: matos@math.univ-lille1.fr.

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properties have been given and recurrence relations between adjacent approximants in a Frobenius–Padé table have been established together with recursive algorithms for their computation.

We are now going to generalize the ideas of Frobenius–Padé approximation to the multivariate case (we will restrict ourselves to the two variable case). Different generalizations of Padé approximation to the multivariate case have been proposed and inspired our work. Different approaches have been developed by Cuyt [7–11], Guillaume [12,13], and other authors (see for instance [17,20,6],...). Properties of the different approximants, convergence properties and ways of computing them have been developed. We take advantage of these different ideas to generalize them to the case of orthogonal expansions.

After giving the general definition of the *multivariate Frobenius–Padé approximants*, we will be interested in the way how to compute them. We will consider two different situations:

- compute the values of a sequence of approximants at a given point (x_0, y_0);
- compute the coefficients of the denominators and numerators of a sequence of approximants.

We will see that these computations are equivalent to the solution of linear systems.

In order to define an approximant we will need to choose three sets of indices: the set of indices appearing in the numerator which we will denote by N, the set of indices appearing in the denominator D and the one corresponding to the terms that will be annihilated in the error term, E. We have also many ways of defining sequences of approximants: fixing the denominator (numerator) and increasing the cardinal of the set of indices in N (respectively, D), increasing simultaneously the sets N and D... . Our aim in this paper is to obtain, for particular choices in this large variety of parameters, either recursive algorithms to compute a sequence of approximations, or a displacement rank structure for the matrix of the system we have to solve to obtain the approximant.

A second generalization of Frobenius–Padé approximation based on the ideas of [12] for Padé approximants is then developed—the *mixed Frobenius–Padé approximants*. We will consider the particular case of a Tchebyshev series for which the computation of the coefficient matrix of the system which gives the denominator coefficients of the approximant is very simple and we will show that the matrix has in this case a block Toeplitz-plus-Hankel structure.

Let us begin first with the definitions.

2. Definition of the approximants

Let us consider a two variable function given by its expansion in an orthogonal series

$$f(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} P_i(x) P_j(y), \tag{1}$$

where

$$\begin{cases} \{P_i\} & \text{is a system of orthogonal polynomials in } [a, b] \text{ with respect to} \\ & \text{the weight function } w, \\ \gamma_{ij} & = \|P_i\| \|P_j\|, \quad \|P\|^2 = \int_a^b P(x)^2 w(x)\,\mathrm{d}x, \\ c_{ij} & = \dfrac{1}{\gamma_{ij}} \int_a^b \int_a^b f(x, y) P_i(x) P_j(y) w(x) w(y)\,\mathrm{d}x\,\mathrm{d}y. \end{cases}$$

We search for two polynomials $P(x, y)$ and $Q(x, y)$

$$\begin{cases} P(x, y) = \sum_{(i,j)\in N} a_{ij} P_i(x) P_j(y), \\ Q(x, y) = \sum_{(i,j)\in D} b_{ij} P_i(x) P_j(y) \end{cases} \tag{2}$$

satisfying

$$f(x, y) Q(x, y) - P(x, y) = \sum_{(i,j)\in(\mathbb{N}^2\setminus E)} d_{ij} P_i(x) P_j(y), \tag{3}$$