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Permanence of a delayed *SIR* epidemic model with density dependent birth rate

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Abstract

In this paper, we consider the permanence of a modified delayed *SIR* epidemic model with density dependent birth rate which is proposed in [M. Song, W. Ma, Asymptotic properties of a revised *SIR* epidemic model with density dependent birth rate and time delay, Dynamic of Continuous, Discrete and Impulsive Systems, 13 (2006) 199–208]. It is shown that global dynamic property of the modified delayed *SIR* epidemic model is very similar as that of the model in [W. Ma, Y. Takeuchi, T. Hara, E. Beretta, Permanence of an *SIR* epidemic model with distributed time delays, Tohoku Math. J. 54 (2002) 581–591; W. Ma, M. Song, Y. Takeuchi, Global stability of an *SIR* epidemic model with time delay, Appl. Math. Lett. 17 (2004) 1141–1145].

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1. Introduction

Epidemic models with or without time delay are studied by many authors (see, for example, for the model with time delay [1,2,11–13,15], for one without time delay [7,9,10,14]). They consider the stability or permanence of the models by applying the theory on delay differential equations [3–6,8]. In this paper, we consider the permanence of the following modified delayed *SIR* epidemic model with density dependent birth rate which is proposed in [13],

$$\begin{cases} \dot{S}(t) = -\beta S(t)I(t-h) - \mu_1 S(t) + b\left(1 - \beta_1 \dfrac{N(t)}{1+N(t)}\right), \\ \dot{I}(t) = \beta S(t)I(t-h) - \mu_2 I(t) - \lambda I(t), \\ \dot{R}(t) = \lambda I(t) - \mu_3 R(t), \end{cases} \tag{1.1}$$

where $S(t)+I(t)+R(t) \equiv N(t)$ denotes the number of a population at time *t*; $S(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible members to the disease, of infective members and of members who have been removed from the possibility

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of infection through full immunity, respectively. It is assumed that all newborns are susceptible. The positive constants μ_1, μ_2 and μ_3 represent the death rates of susceptibles, infectives and recovered, respectively. It is natural biologically to assume that $\mu_1 \leqslant \min\{\mu_2, \mu_3\}$. The positive constants b and λ represent the birth rate of the population and the recovery rate of infectives, respectively. The constant β_1 $(0 \leqslant \beta_1 < 1)$ reflects the relation between the birth rate and the density of population. The nonnegative constant h is the time delay.

The initial condition of (1.1) is given as

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta) \quad (-h \leqslant \theta \leqslant 0), \tag{1.2}$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)^{\mathrm{T}} \in C$, such that $\varphi_i(\theta) \geqslant 0$ $(-h \leqslant \theta \leqslant 0,\ i = 1, 2, 3)$. C denotes the Banach space $C([-h, 0], \mathscr{R}^3)$ of continuous functions mapping the interval $[-h, 0]$ into $\mathscr{R}^3$. By a biological meaning, we further assume that $\varphi_i(0) > 0$ for $i = 1, 2, 3$. It is easily to show that the solution $(S(t), I(t), R(t))$ of (1.1) with the initial condition (1.2) exists for all $t \geqslant 0$ and is unique and positive for all $t \geqslant 0$.

With some simple computation, we see that (1.1) always has a disease free equilibrium (i.e., boundary equilibrium) $E_0 = (S_0, 0, 0)$, where

$$S_0 = \frac{1}{2\mu_1}\left[b(1-\beta_1) - \mu_1 + \sqrt{[b(1-\beta_1) - \mu_1]^2 + 4\mu_1 b}\right].$$

Furthermore, if

$$S_0 > S^* \equiv \frac{\mu_2 + \lambda}{\beta}, \tag{1.3}$$

then (1.1) also has an endemic equilibrium (i.e., interior equilibrium) $E_+ = (S^*, I^*, R^*)$, where

$$I^* = -P + \frac{\sqrt{P^2 - 4\beta S^* W Q}}{2\beta S^* W}, \quad R^* = \frac{\lambda I^*}{\mu_3},$$

$$W = 1 + \frac{\lambda}{\mu_3} > 0,$$

$$P = [\mu_1 S^* - b(1-\beta_1)]W + \beta S^*(1 + S^*),$$

$$Q = [\mu_1 S^* - b(1-\beta_1)](1 + S^*) - b\beta_1 < 0.$$

A detailed analysis on the local asymptotic stability of E_0 and E_+, and the global asymptotic stability of E_0 are given in [13]. The purpose of the paper is to consider the permanence of (1.1) with the initial condition (1.2).

2. Permanence of (1.1)

In this section, we always assume that $S_0 > S^*$ which ensures the existence of the endemic equilibrium E_+ of (1.1). The following lemma is proved in [13].

Lemma 2.1. *For any solution* $(S(t), I(t), R(t))$ *of* (1.1) *with* (1.2), *we have that*

$$\limsup_{t \to +\infty} N(t) \leqslant S_0. \tag{2.1}$$

We also have the following

Lemma 2.2. *For any solution* $(S(t), I(t), R(t))$ *of* (1.1) *with* (1.2), *it has that*

$$\liminf_{t \to +\infty} S(t) \geqslant \frac{b(1-\beta_1)}{\beta S_0 + \mu_1} \equiv v_1. \tag{2.2}$$