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ScienceDirect
Journal of Computational and Applied Mathematics 200 (2007) 21–31
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Error estimates of the DtN finite element method for the exterior Helmholtz problem

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Received 13 September 2004; received in revised form 7 December 2005

Abstract

A priori error estimates are established for the DtN (Dirichlet-to-Neumann) finite element method applied to the exterior Helmholtz problem. The error estimates include the effect of truncation of the DtN boundary condition as well as that of the finite element discretization. A property of the Hankel functions which plays an important role in the proof of the error estimates is introduced.

Keywords: Exterior Helmholtz problem; Artificial boundary condition; Drichlet-to-Neumann operator; Finite element method; A priori error estimate

1. Introduction

We consider the exterior Helmholtz problem:

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \gamma, \\ \lim_{r \to +\infty} r^{(d-1)/2} \left(\dfrac{\partial u}{\partial r} - \mathrm{i} k u \right) = 0 & \text{(the outgoing radiation condition)}, \end{cases} \tag{1}$$

where k, called the wave number, is a positive constant, Ω is an unbounded domain of $\mathbb{R}^d$ ($d = 2$ or 3) with sufficiently smooth boundary γ, f is a given datum, $r = |x|$ for $x \in \mathbb{R}^d$, and $\mathrm{i} = \sqrt{-1}$. Assume that $\mathcal{O} \equiv \mathbb{R}^d \backslash \overline{\Omega}$ is a bounded open set and that f has a compact support. Problem (1) arises in models of acoustic scattering by a sound-soft obstacle $\mathcal{O}$ embedded in a homogeneous medium.

To solve numerically problem (1), one often introduces an artificial boundary in order to reduce the computational domain to a bounded domain and imposes an artificial boundary condition on the artificial boundary. Although a variety of artificial boundary conditions have been proposed (see, e.g., [7], for a review), we focus on the exact nonlocal boundary condition based on the *Dirichlet-to-Neumann* (DtN) operator, which is called the exact DtN boundary condition. The comparison of the exact DtN boundary condition with local artificial boundary conditions is described in [8,10]. Imposing the exact DtN boundary condition on the artificial boundary, we can reduce problem (1) equivalently to a problem on the bounded domain between the artificial boundary and the boundary γ. We discretize the reduced problem by using the finite element method. This method is called the DtN finite element method.

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doi:10.1016/j.cam.2005.12.004

The DtN finite element method for the exterior Helmholtz problem has first been proposed by MacCamy–Marin [23] in 1980, who represented the DtN operator through an integral equation. Feng [4], Masmoudi [24], and Keller–Givoli [18] have derived the Fourier series representation of the DtN operator. Masmoudi [24] and Keller–Givoli [18] incorporated such a representation directly into the finite element method.

The DtN finite element method for other kinds of problems concerning the Helmholtz equation has been investigated by several authors. In 1978, Fix–Marin [6], who are pioneers in the DtN finite element method, studied the under-water acoustic problem. Goldstein [11] established error estimates for the Helmholtz problem on unbounded waveguides. Bao [1] also established error estimates for the problem concerning the diffraction of a time harmonic wave incident on a periodic surface of some inhomogeneous material.

In China, independently of the western world, the DtN finite element method has been suggested and developed first by Feng and Yu in 1980 and 1982, for example, see [5], where they called it the canonical boundary element method or the natural boundary element method. Yu has published many papers [31,32,30,35,17] and a monograph [33,34] (see also its review [9]) in this direction. In Ushijima's paper [28], we can find these words: "Chinese scholars, Feng Kang, Han Houde, Yu De-hao and others should be mentioned among founders of the treatment. In western world, J.B. Keller and D. Givoli are also should be quoted."

In this paper, we establish a priori error estimates in the H^1- and L^2-norms for the exterior Helmholtz problem. The use of the Fourier series representation of the DtN operator requires truncating the series in practical computations. So we analyze the series truncation error as well as the finite element discretization error. To the best knowledge of the author, our error estimates are new, because no error estimate treating both the truncation error and the discretization error simultaneously has been published yet (cf. [10, p. 32]). MacCamy–Marin [23] and Masmoudi [24] have derived an error estimate, but their estimate depends only on the mesh size (see also [3,17]). The counterpart of our error estimates for the Helmholtz problem on unbounded waveguides has been established by Goldstein [11] in 1982. Our error analysis roughly follows his analysis; however, we need some properties of the Hankel functions, which contain a new and important result (Lemma 4); we were inspired to prove Lemma 4 by Han–Bao [12, Lemma 3.1]. We here remark that in the error analysis of ours (and also of Goldstein), the argument of Schatz [27] plays an essential role, since the Helmholtz equation is indefinite.

Analysis of the truncation error in the DtN finite element method is an important topic. For problems of the positive definite type, Yu [32] and Han–Wu [13] have first derived error estimates including the truncation error for the exterior Laplace problem in 1985. After that, such error estimates for other problems are established in many papers, for example, in Han–Wu [14] for the linear elastic problem, and in Givoli–Patlashenko–Keller [10] and Han–Bao [12] for a certain class of the linear elliptic second order boundary value problem on exterior domains and semi-infinite strips (the error estimate in [12] is more sophisticated than that in [10]). For problems of other types, we mention Lenoir–Tounsi [21] (the sea-keeping problem) and Koyama–Tanimoto–Ushijima [20] (the eigenvalue problem of the linear water wave in a water region with a reentrant corner).

The remainder of this paper is organized as follows. In Section 2, we formulate the reduced problem with the DtN boundary condition. In Section 3, we introduce some properties of the Hankel functions, which are employed for establishing the error estimates in Section 4.

2. The DtN formulation

We first introduce a theorem concerning the well-posedness of problem (1).

Theorem 1. *For every compactly supported* $f \in L^2(\Omega)$, *problem* (1) *has a unique solution in* $H^2_{\mathrm{loc}}(\overline{\Omega})$, *where*

$$H^m_{\mathrm{loc}}(\overline{\Omega}) = \{u | u \in H^m(B) \text{ for all bounded open set } B \subset \Omega\} \quad (m \in \mathbb{N}).$$

Here $L^2(\Omega)$ *denotes the usual space of complex-valued square integrable functions on* Ω, *and* $H^m(B)$ *denotes the usual complex Sobolev space on* B (*see, e.g.,* [22]).

Proof. See [25,26]. □