Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 200 (2007) 226 – 230
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Short Communication

On multiple roots in Descartes' Rule and their distance to roots of higher derivatives

Arno Eigenwillig*

Max-Planck-Institut für Informatik, 66123 Saarbrücken, Germany

Received 11 August 2005; received in revised form 19 December 2005

Abstract

If an open interval I contains a k-fold root α of a real polynomial f, then, after transforming I to $(0, \infty)$, Descartes' Rule of Signs counts exactly k roots of f in I, provided I is such that Descartes' Rule counts no roots of the kth derivative of f. We give a simple proof using the Bernstein basis.

The above condition on I holds if its width does not exceed the minimum distance σ from α to any complex root of the kth derivative. We relate σ to the minimum distance s from α to any other complex root of f using Szegő's composition theorem. For integer polynomials, $\log(1/\sigma)$ obeys the same asymptotic worst-case bound as $\log(1/s)$.

MSC: Primary: 26C10; 30C15; 65H05

Keywords: Descartes' Rule of Signs; Descartes–Jacobi Rule; Bernstein basis; Root isolation; Root separation

1. Introduction

Let $f(x) = \sum_{i=0}^{n} f_i x^i$ be a polynomial of degree n with real coefficients. *Descartes' Rule of Signs* states that the number $v = \mathrm{var}(f_0, \ldots, f_n)$ of sign variations in the coefficient sequence of f exceeds the number p of positive real roots, counted with multiplicities, by an even non-negative integer. See [7, Theorem 2] for a proof with careful historic references. Jacobi [5, IV] made the "little observation" that this statement on the interval $(0, \infty)$ can be extended to any open interval (l, r) by first composing f with the Möbius transform $T(x) = (lx + r)/(x + 1)$ that takes $(0, \infty)$ to (l, r) and then inspecting the coefficients of $g(x) = (x + 1)^n f(T(x))$. We call this the *Descartes test* for the number of roots of f in (l, r). As this test counts roots with multiplicities, it cannot distinguish, say, two simple roots from one double root. However, if multiplicities are known in advance, the Descartes test remains useful even in the presence of multiple roots.

Consider the *Bernstein basis* $B_0^n, B_1^n, \ldots, B_n^n$ defined by

$$B_i^n(x) = B_i^n[l, r](x) = \binom{n}{i} \frac{(x - l)^i (r - x)^{n-i}}{(r - l)^n}. \qquad (1)$$

* Tel.: +49 681 9325 129; fax: +49 681 9325 199.
E-mail address: arno@mpi-inf.mpg.de.

0377-0427/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.cam.2005.12.016

The coefficients of $f(x) = \sum_{i=0}^{n} b_i B_i^n(x)$ in the Bernstein basis and the coefficients of $g(x) = \sum_{i=0}^{n} g_i x^i$ in the monomial basis agree up to order and positive constants; that is, $g_i = \binom{n}{i} b_{n-i}$ for $0 \leqslant i \leqslant n$. Hence the Descartes test for f on (l, r) can equivalently be formulated to count the number of sign variations in the sequence $(b_0, \ldots, b_n)$ of Bernstein coefficients. This relation was already known to Pólya and Schoenberg [11, p. 322; 14, Section 1]. It was first applied in a root isolation algorithm by Lane and Riesenfeld [8].

Descartes' Rule of Signs has an immediate geometric interpretation in terms of Bézier curves [4,12]. The graph of f is a Bézier curve with control points $(\mathbf{b}_i)_{i=0}^{n}$ where $\mathbf{b}_i = (i/n, b_i)$. The Descartes test counts how many times the control polygon $\overline{\mathbf{b}_0\mathbf{b}_1} \cup \overline{\mathbf{b}_1\mathbf{b}_2} \cup \cdots \cup \overline{\mathbf{b}_{n-1}\mathbf{b}_n}$ crosses the x-axis. Repeated de Casteljau subdivision is a corner-cutting process on the control polygon, which, in the limit, converges to the graph of f. The number of intersections with the x-axis can never grow and only drop by an even number during corner-cutting.

Let the open interval I contain a simple root α of f. If the Descartes test counts $v = 1$ in I, this implies that α is the unique root of f in I. However, the converse implication does not hold in general. The use of the Descartes test in algorithms for isolating the real roots of square-free polynomials [2,6,13,9] has motivated research on conditions sufficient for the Descartes test to count $v = 1$. A particularly general sufficient condition was given by Ostrowski [10] but has been overlooked until recently [7].

We remark that not every test for roots in an interval I has this property of yielding the exact count if the width of I is small enough. The closely related Budan–Fourier test for roots in $(l, r]$ computes $v' := \mathrm{var}(f(l), f'(l), \ldots, f^{(n)}(l)) - \mathrm{var}(f(r), f'(r), \ldots, f^{(n)}(r))$, which is also known to exceed the number of roots in $(l, r]$, counted with multiplicities, by a non-negative even integer [1, Theorem 2.36]. But consider the example $f(x) = x^3 + x$: Substitution into the sequence $(f(x), \ldots, f'''(x))$ yields the sign patterns $(-, +, -, +)$ for $x < 0$ and $(+, +, +, +)$ for $x > 0$. Hence $v' = 3$ for any interval $(l, r]$ containing the simple root 0 in its interior.

Our results: Let the open interval I contain a k-fold root α of f whose multiplicity $k \geqslant 1$ is known. We present sufficient conditions for the Descartes test to count $v = k$ sign variations for f in I and thus to certify uniqueness of the root α in I. Using the Bernstein basis, we can prove very easily in Section 2 that the Descartes test counts $v = k$ for f whenever it counts $v^{(k)} = 0$ for the kth derivative $f^{(k)}$. This condition is met if the width of I does not exceed the minimum distance σ between α and any root of $f^{(k)}$. In Section 3, we relate σ to the minimum distance s between α and any other complex root of f. To do so, we use Szegő's composition theorem in a way that generalizes an approach of Dimitrov [3] from the first to higher derivatives of f, and we obtain a lower bound on the distance of α to roots of $f^{(r)}$ for any $r \leqslant k$. For integer polynomials with τ-bit coefficients, the resulting bound on $\log(1/\sigma)$ has the same worst-case asymptotics as $\log(1/s)$, namely $O(n\tau + n \log n)$.

2. A partial converse via differentiation

From now on, we assume w.l.o.g. $[l, r] = [0, 1]$. Differentiation of the ith Bernstein basis polynomial $B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}$ yields $nB_{i-1}^{n-1}(x) - nB_i^{n-1}(x)$, where $B_{-1}^{n-1} = B_n^{n-1} = 0$ by convention. Hence the derivative of $f(x) = \sum_{i=0}^{n} b_i B_i^n(x)$ is $f'(x) = \sum_{i=0}^{n} n(b_{i+1} - b_i) B_i^{n-1}(x)$. The coefficient vector $(c_i)_{i=0}^{n-1}$ of $1/n \cdot f'(x)$ is therefore given by the following difference scheme.

$$\begin{array}{ccccc} b_0 & b_1 & b_2 \cdots b_{n-1} & & b_n \\ c_0 = -b_0 + b_1 & c_1 = -b_1 + b_2 & \cdots & c_{n-1} = -b_{n-1} + b_n & \end{array} \tag{2}$$

The following lemma can be regarded as a piecewise linear analogue of Rolle's Theorem.

Lemma 1. *The numbers of sign variations in* (2) *satisfy* $\mathrm{var}(c_0, \ldots, c_{n-1}) \geqslant \mathrm{var}(b_0, \ldots, b_n) - 1$.

Proof. Each sign variation in $(b_0, \ldots, b_n)$ is an index pair $0 \leqslant i < j \leqslant n$ such that $b_i b_j < 0$ and $b_{i+1} = \cdots = b_{j-1} = 0$. Let there be exactly v such pairs $(i_1, j_1), \ldots, (i_v, j_v)$ with indices $i_1 < j_1 \leqslant i_2 < j_2 \leqslant \cdots \leqslant i_v < j_v$. Sign variations are either "positive to negative" ($b_i > 0$) or "negative to positive" ($b_i < 0$). Obviously, these types alternate. If $b_{i_\ell} > 0$, then $b_{i_\ell + 1} \leqslant 0$ and thus $c_{i_\ell} = -b_{i_\ell} + b_{i_\ell + 1} < 0$. Similarly, if $b_{i_\ell} < 0$ then $c_{i_\ell} > 0$. Hence the sequence $(c_0, \ldots, c_{n-1})$ contains an alternating subsequence $\mathrm{sgn}(c_{i_1}) \neq \mathrm{sgn}(c_{i_2}) \neq \cdots \neq \mathrm{sgn}(c_{i_v})$, demonstrating that $(c_0, \ldots, c_{n-1})$ has at least $v - 1$ sign variations. □