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A note on the bounds of the error of Gauss–Turán-type quadratures☆

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Abstract

This note is concerned with estimates for the remainder term of the Gauss–Turán quadrature formula,

$$R_{n,s}(f) = \int_{-1}^{1} w(t) f(t)\,\mathrm{d}t - \sum_{v=1}^{n} \sum_{i=0}^{2s} A_{i,v} f^{(i)}(\tau_v),$$

where $w(t) = (U_{n-1}(t)/n)^2\sqrt{1-t^2}$ is the Gori–Michelli weight function, with $U_{n-1}(t)$ denoting the $(n-1)$th degree Chebyshev polynomial of the second kind, and f is a function analytic in the interior of and continuous on the boundary of an ellipse with foci at the points ± 1 and sum of semiaxes $\varrho > 1$. The present paper generalizes the results in [G.V. Milovanović, M.M. Spalević, Bounds of the error of Gauss–Turán-type quadratures, J. Comput. Appl. Math. 178 (2005) 333–346], which is concerned with the same problem when $s = 1$.

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1. Introduction

Let w be an integrable weight function on the interval $(-1, 1)$. We consider the error term $R_{n,s}(f)$ of the Gauss–Turán quadrature formula with multiple nodes

$$\int_{-1}^{1} w(t) f(t)\,\mathrm{d}t = \sum_{v=1}^{n} \sum_{i=0}^{2s} A_{i,v} f^{(i)}(\tau_v) + R_{n,s}(f),$$

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which is exact for all algebraic polynomials of degree at most $2(s+1)n-1$, and whose nodes are the zeros of the corresponding s-orthogonal polynomial $\pi_{n,s}(t)$ of degree n. For more details on Gauss–Turán quadratures and s-orthogonal polynomials see the book [1] and the survey paper [4].

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and D be its interior. If the integrand f is an analytic function in D and continuous on $\overline{D}$, then we take as our starting point the well-known expression of the remainder term $R_{n,s}(f)$ in the form of the contour integral

$$R_{n,s}(f)=\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma} K_{n,s}(z)f(z)\,\mathrm{d}z. \tag{1.1}$$

The kernel is given by

$$K_{n,s}(z)=\frac{\varrho_{n,s}(z)}{[\pi_{n,s}(z)]^{2s+1}},\quad z\notin[-1,1], \tag{1.2}$$

where

$$\varrho_{n,s}(z)=\int_{-1}^{1}\frac{[\pi_{n,s}(t)]^{2s+1}}{z-t}w(t)\,\mathrm{d}t,\quad n\in\mathbb{N}, \tag{1.3}$$

and $\pi_{n,s}(t)$ is the corresponding s-orthogonal polynomial with respect to the weight function $w(t)$ on $(-1,1)$.

The integral representation (1.1) leads to a general error estimate, by using Hölder inequality,

$$|R_{n,s}(f)|=\frac{1}{2\pi}\left|\oint_{\Gamma} K_{n,s}(z)f(z)\,\mathrm{d}z\right|\leqslant\frac{1}{2\pi}\left(\oint_{\Gamma}|K_{n,s}(z)|^{r}|\mathrm{d}z|\right)^{1/r}\left(\oint_{\Gamma}|f(z)|^{r'}|\mathrm{d}z|\right)^{1/r'},$$

i.e.,

$$|R_{n,s}(f)|\leqslant\frac{1}{2\pi}\|K_{n,s}\|_{r}\|f\|_{r'}, \tag{1.4}$$

where $1\leqslant r\leqslant+\infty$, $1/r+1/r'=1$, and

$$\|f\|_{r}:=\begin{cases}\left(\oint_{\Gamma}|f(z)|^{r}|\mathrm{d}z|\right)^{1/r}, & 1\leqslant r<+\infty,\\ \max\limits_{z\in\Gamma}|f(z)|, & r=+\infty.\end{cases}$$

The case $r=+\infty$ ($r'=1$) gives

$$|R_{n,s}(f)|\leqslant\frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{1.5}$$

where $\ell(\Gamma)$ is the length of the contour Γ. On the other side, for $r=1$ ($r'=+\infty$), the estimate (1.4) reduces to

$$|R_{n,s}(f)|\leqslant\frac{1}{2\pi}\left(\oint_{\Gamma}|K_{n,s}(z)||\mathrm{d}z|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{1.6}$$

which is evidently stronger than the previous, because of inequality

$$\oint_{\Gamma}|K_{n,s}(z)||\mathrm{d}z|\leqslant\ell(\Gamma)\left(\max_{z\in\Gamma}|K_{n,s}(z)|\right).$$

Also, the case $r=r'=2$ could be of certain interest.

For getting the estimate (1.5) or (1.6) it is necessary to study the magnitude of $|K_{n,s}(z)|$ on Γ or the quantity

$$L_{n,s}(\Gamma):=\frac{1}{2\pi}\oint_{\Gamma}|K_{n,s}(z)|\,|\mathrm{d}z|,$$

respectively (see, e.g., [5,6]).