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A two-stage method for nonlinear inverse problems☆

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Abstract

In this work we are interested in the solution of nonlinear inverse problems of the form $F(x)=y$. We consider a two-stage method which is third order convergent for well-posed problems. Combining the method with Levenberg–Marquardt regularization of the linearized problems at each stage and using the discrepancy principle as a stopping criterion, we obtain a regularization method for ill-posed problems. Numerical experiments on some parameter identification and inverse acoustic scattering problems are presented to illustrate the performance of the method.

MSC: 65J15; 65J20; 65J22

Keywords: Nonlinear ill-posed problems; Regularization methods; Parameter identification; Inverse acoustic scattering

1. Introduction

Consider nonlinear inverse problems of the form

$$F(x) = y, \tag{1}$$

where $F : D(F) \subseteq X \to Y$ is a nonlinear differentiable operator between Hilbert spaces X and Y. Suppose that (1) has at least one solution $\hat{x}$. Usually, in inverse problems, the map F is compact and (1) is an ill-posed problem, in the sense that the presence of small noise in the datum y can yield drastic changes in the solution. On the other hand, almost always only noisy data are available in practice, so that the numerical solution of (1) under ill-posedness assumption is a very difficult task. Some regularization technique has to be applied to compute reasonable approximations of a solution in a stable manner.

Several methods for nonlinear ill-posed problems of form (1) have been studied in the last decade (see e.g. [1,5–9,12,15,16]). Most of them are based on the least-squares reformulation.

As it is well known from the theory of numerical methods for well-posed nonlinear problems [4], algorithms exploiting second order information can be fruitfully used to solve problems which exhibit high nonlinearity and/or large residual. This is the case for many classical inverse problems. Recently, Hettlich and Rundell [10] proposed a predictor–corrector method involving the second Fréchet derivative of F. This method is of third order for well-posed

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problems; it shows very good theoretical properties for ill-posed problems as well, when Tikhonov regularization is applied to both prediction and correction steps with regularization parameters constant throughout the iterations. In fact, for solving problems where it is difficult to (cheaply) evaluate F'' at all iterates, the authors suggested a "frozen" version of their method, where the derivatives computed at the initial guess are used at all subsequent iterations.

In this paper, we consider the following two-stage iterative process, that is third order for well-posed problems [14], and avoids the computation of F'' by evaluating F' at suitable points, in an analogous spirit to the second order Runge–Kutta methods for ordinary differential equations:

$$\begin{aligned} &F'(x_n)\hat{h}_n = y - F(x_n), \\ &z_n = x_n + \tfrac{1}{2}\hat{h}_n, \\ &F'(z_n)h_n = y - F(x_n), \\ &x_{n+1} = x_n + h_n. \end{aligned} \tag{2}$$

Our aim is to use this method for solving ill-posed problems, in which case the linearized problems are in general ill-posed as well, and some regularization tool has to be used. Suppose to know noisy data y^δ such that $\|y^\delta - y\| \leqslant \delta$ for a given noise level $\delta \geqslant 0$. By applying Levenberg–Marquardt regularization to both linearized problems in (2), with fixed parameters $\alpha_1 > 0$ for the first stage and $\alpha_2 > 0$ for the second one, we obtain the following regularized scheme:

$$\begin{aligned} &(F'(x_n^\delta)^* F'(x_n^\delta) + \alpha_1 I)\hat{h}_n^\delta = F'(x_n^\delta)^*(y^\delta - F(x_n^\delta)), \\ &z_n^\delta = x_n^\delta + \tfrac{1}{2}\hat{h}_n^\delta, \\ &(F'(z_n^\delta)^* F'(z_n^\delta) + \alpha_2 I)h_n^\delta = F'(z_n^\delta)^*(y^\delta - F(x_n^\delta)), \\ &x_{n+1}^\delta = x_n^\delta + h_n^\delta, \end{aligned} \tag{3}$$

where the superscript δ is used to remark the dependency of the iterations on δ.

In the next section, we prove that this method combined with a suitable stopping criterion is a regularization method. In Section 3, we present numerical results on some classical parameter identification problems and on an inverse acoustic scattering problem. The experiments show very good performance compared to the Newton's method, from the point of view of the robustness and of the quality of the reached reconstructions.

2. Convergence analysis

In the sequel, $B(z, r)$ will denote the closed ball $\{x \in X : \|x - z\| \leqslant r\}$. The convergence analysis follows essentially the scheme of [7] and requires that F satisfies the following assumptions, where $\Omega \subset D(F)$ is an open neighborhood of $\hat{x}$:

A1. there exists $M > 0$ such that $\|F'(x)\| \leqslant M$ for $x \in \Omega$;
A2. F' is uniformly Lipschitzian in Ω with Lipschitz constant L_1;
A3. there exists $C > 0$ such that

$$\|F(z) - F(x) - F'(x)(z - x)\| \leqslant C\|z - x\|\|F(z) - F(x)\| \quad \forall x, z \in \Omega.$$

Under these assumptions, and assuming x_0 close enough to $\hat{x}$, we will prove that the sequence $\{x_n\}$ converges to a solution x^* of (1) when $\delta = 0$. On the other hand, for $\delta > 0$, we will show that the condition

$$\|y^\delta - F(x_n^\delta)\| \leqslant \tau\delta \tag{4}$$

for a fixed $\tau > 1$, can be satisfied within a finite number of iterations. So, in practice, we will stop the iterative process when (4) is fulfilled for the first time. This is the well known discrepancy principle and τ works as an additional regularization parameter.

We show first some monotonicity results for the errors sequence.