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Bifurcation diagrams of population models with nonlinear diffusion☆

Young He Lee[a], Lena Sherbakov[a], Jackie Taber[a], Junping Shi[a,b,*]

[a]Department of Mathematics, The College of William and Mary, Williamsburg, VA 23187, USA
[b]School of Mathematics, Harbin Normal University, Harbin, Heilongjiang, 150080, PR China

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Abstract

We develop analytical and numerical tools for the equilibrium solutions of a class of reaction–diffusion models with nonlinear diffusion rates. Such equations arise from population biology and material sciences. We obtain global bifurcation diagrams for various nonlinear diffusion functions and several growth rate functions.

MSC: 35J65; 35B32

Keywords: Global bifurcation; Semilinear elliptic equation; Nonlinear diffusion

1. Introduction

Diffusion mechanism models the movement of many individuals in an environment or media. The individuals can be very small such as basic particles in physics, bacteria, molecules, or cells, or very large objects such as animals, plants, or certain kind of events like epidemics, or rumors. By using the random

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* Corresponding address. Department of Mathematics, The College of William and Mary, Williamsburg, VA 23187, USA. Tel.: +1 7572212030; fax: +1 7572217400.

E-mail address: shij@math.wm.edu (J. Shi).

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walk or Fick's law, one can derive a one-dimensional reaction–diffusion model (see [9,10,20]):

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\,\frac{\partial u}{\partial x}\right) + f(u), \tag{1.1}$$

where $u(x,t)$ is the density function of the organism on a one-dimensional spatial domain, the diffusion rate D is a constant, and $f(u)$ is the growth rate. However, in some situations, the random walk can be biased and the diffusion rate can depend on the density of the population. In [20,19], Turchin derives a partial differential equation model with nonlinear diffusion:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u)\,\frac{\partial u}{\partial x}\right) + f(u), \tag{1.2}$$

where $D(u)$ is a positive quadratic function; and another model of animal dispersal is also of form (1.2) with $D(u) = u^m$ for some $m > 0$ (see [9,10]). Such model also appears as the porous media equation (with $D(u) = u^m$ again) in material science (see [4]).

In this paper, we use analytic and numerical tools to consider the equilibrium solutions of (1.2) with Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$. These conditions are appropriate for investigating species that are bound to their habitat (i.e. if they leave outside of their boundary, they will die off immediately). After a nondimensionalization scaling, we consider the equation

$$[D(u)u']' + \lambda f(u) = 0, \quad u(0) = u(1) = 0, \tag{1.3}$$

where $D(u)$ is a nonnegative smooth function defined on $\mathscr{R}^+$, and λ is a positive parameter. Note that if $D(u)$ is now a dimensionless diffusion function, then $\lambda = L^2/D$, where L is the length of the interval, and D is a scale of the diffusion rate. Thus a larger λ is equivalent to larger habitat size and slower diffusion.

For the nonlinear growth rate $f(u)$, we will consider three different growth patterns: (a) logistic growth; (b) weak Allee effect; and (c) strong Allee effect. In general, the logistic growth is characterized by a non-increasing growth rate per capita $f(u)/u$, and the Allee effect is when the growth rate per capita changes from increasing to decreasing as the population density increases. In the latter case, if the growth rate is positive at zero population, then it is called weak Allee effect, and if negative, then it is strong Allee effect. A more detailed discussion has been given in [17]. In this paper, for the sake of simplicity, we will only consider the representing examples of each case, (a) logistic $f(u) = u(1-u)$; (b) weak Allee effect $f(u) = ku(1-u)(u+b)$ for some $k > 0$ and $b \in (0,1)$; and (c) strong Allee effect $f(u) = ku(1-u)(u-b)$ for some $k > 0$ and $b \in (0,1)$.

Following earlier work by Opial [11] and Laetsch [5] for the case of $D(u) \equiv 1$ (i.e. linear diffusion case), we develop analytic formulas for the bifurcation diagrams of positive solutions to (1.3). These formulas are generalizations of well-known time-mapping first developed in [11] which is used to calculate the periods of nonlinear oscillators when $D(u)$ is a constant function. The bifurcation diagrams of (1.3) when $D(u) \equiv 1$ have been considered in [11,5,18,8,7,13,21,6,22], and Schaaf [13] also briefly considers the case of nonlinear $D(u)$ but different situations. Cantrell and Cosner [1–3] and Shi and Shivaji [17] study the equilibrium solutions of (1.3) in a more general setting, but their methods are quite different and our results here are more specific. An alternative approach to the bifurcation diagram is to use a transformation $v = \int D(u)\,\mathrm{d}u$, and to consider the equation $v'' + \lambda f(u^{-1}(v)) = 0$ (see [17]), but practically the inverse of v is often difficult to calculate, and our approach here is more direct. The derivation of the formulas are given in Section 2, and some analytic results on the monotonicity of the diagrams are also given in