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Eigenfrequencies of fractal drums

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Abstract

A method for the computation of eigenfrequencies and eigenmodes of fractal drums is presented. The approach involves first conformally mapping the unit disk to a polygon approximating the fractal and then solving a weighted eigenvalue problem on the unit disk by a spectral collocation method. The numerical computation of the complicated conformal mapping was made feasible by the use of the fast multipole method as described in [L. Banjai, L.N. Trefethen, A multipole method for Schwarz–Christoffel mapping of polygons with thousands of sides, SIAM J. Sci. Comput. 25(3) (2003) 1042–1065]. The linear system arising from the spectral discretization is large and dense. To circumvent this problem we devise a fast method for the inversion of such a system. Consequently, the eigenvalue problem is solved iteratively. We obtain eight digits for the first eigenvalue of the Koch snowflake and at least five digits for eigenvalues up to the 20th. Numerical results for two more fractals are shown.

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1. Introduction

Objects in nature are not always well represented by simple geometries such as circles or straight lines. As evidence for this Mandelbrot [24,25] used the experiments by Richardson to show that some coastlines are better modelled by curves of infinite length than by compositions of smooth curves. The overwhelming evidence that objects in nature can be modelled by fractals leads to the question of how physical processes on fractals can be described. Also some physical processes seem to generate fractals [37].

One particular physical process that has attracted theoretical [19,21], experimental [31], and numerical investigation [15,20] is the mechanical vibration of a fractal drum. The hope is that research in this direction might shed light on such problems as the dependence of sea waves on topography of the coastline. The mere existence of fractal coastlines suggests good damping properties of fractal shapes [30].

In this paper, we develop a numerical method for the computation of the eigenvalues and eigenfunctions of the Dirichlet Laplacian on fractal domains. We approximate a fractal with a polygon of many thousands of vertices and solve the eigenvalue problem on this polygon. Numerical solution of eigenvalue problems on polygons is a classical problem, see [13], and has recently been very successfully solved for polygons with few vertices [3,7]. However, all of

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these methods become too expensive once the number of vertices of the polygon runs into thousands. Hence alternative methods are required when the domain of interest is a fractal.

As our main example we study the steady-state vibrations of a Koch snowflake drum. Computations of eigenvalues and eigenfunctions of such a system have already been done by using finite differences on polygonal approximations to the fractal domain [20]. Using their numerical results Lapidus et al. have produced beautiful images of eigenmodes; these images have subsequently been realized as mathematical sculptures by the artist Helaman Ferguson [10]. Recently, a different grid for the finite differences has been used to obtain more accurate results [27].

Our method consists of transplantation from a Koch snowflake polygon to the unit disk and then the solution of the modified eigenvalue problem on the unit disk by a spectral collocation method. The idea of using conformal mapping to simplify the computational domain is by no means new. The more common approach to solving Poisson equations is to map the domain onto a rectangle and then use finite difference or finite element discretizations on the rectangle; for a review of this and many other applications of conformal mapping see [32]. Cureton and Kuttler [5] have computed eigenvalues of the hexagon by conformally mapping the domain to the unit disk and then applying the Rayleigh–Ritz method, with the eigenfunctions of the unweighted problem on the disk as the trial functions. Mason uses a conformal map to straighten the reentrant corner of the *L*-shaped membrane and then applies a spectral method to the transplanted problem [26].

In numerical experiments we find that, whereas previous studies have achieved around three [20] or four [27] digits of accuracy, this method appears to provide at least five and up to eight digits, depending on the eigenvalue, not only for the approximating polygons but also for the Koch snowflake fractal. We do not prove that our results are this accurate, but the experimental evidence is compelling. We also compute eigenvalues for further two fractals. One is an example of a fractal for which our method works even better than for the Koch snowflake whereas for the second it performs less well. We give reasons why such a behaviour is to be expected.

2. Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain. We consider the situation where a homogeneous membrane is stretched and then fixed along its boundary $\partial\Omega$ and the tension per unit length caused by stretching is the same at all points and all directions and does not change during motion. Let $U(x, y, t)$ be a function that gives the displacement of the membrane at point $(x, y) \in \mathbb{R}^2$ and at time $t \geqslant 0$. Then U satisfies the wave equation

$$U_{tt} = \Delta U, \tag{1}$$

with boundary condition

$$U(x_0, y_0, t) = 0, \quad (x_0, y_0) \in \partial\Omega. \tag{2}$$

By separation of variables $U(x, y, t) = u(x, y)w(t)$ the wave equation gives

$$w''(t) + \lambda w(t) = 0,$$

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega,$$

where $\lambda > 0$ is a constant. Since the first equation is trivial we concentrate on finding the eigenvalue-eigenfunction pairs (λ, u) such that

$$\begin{aligned} &\Delta u + \lambda u = 0 \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3}$$

It is known that the spectrum of the Dirichlet Laplacian is discrete and consists of an infinite sequence of eigenvalues $\{\lambda_i\}$

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$$

such that $\lambda_n \to \infty$ as $n \to \infty$; see [6]. The corresponding eigenfunctions u_n are infinitely differentiable in Ω and u_1 can be chosen so that $u_1 > 0$.