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Short communication

Superconvergence of the continuous Galerkin finite element method for delay-differential equation with several terms

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Abstract

The continuous Galerkin finite element method for linear delay-differential equation with several terms is studied. Adding some lower terms in the remainder of orthogonal expansion in an element so that the remainder satisfies more orthogonal condition in the element, and obtain a desired superclose function to finite element solution, thus the superconvergence of p-degree finite element approximate solution on $(p+1)$-order Lobatto points is derived.

MSC: 65M60

Keywords: Delay-differential equation with several terms; Continuous Galerkin finite element method; New element orthogonal expansion; Superconvergence

1. Introduction

Delay-differential equations (DDEs) are a large and important class of dynamical systems. There are different kinds of delay-differential equations. They often arise in either natural or technological control problems. The investigations recorded in most of references were concentrated on Runge–Kutta method or some multi-step methods [7,8,10]. In this paper we will study the continuous Galerkin finite element method for a class of linear delay-differential equation

$$u'(t) = a(t)u(t) + \sum_{l=1}^{m} b_l(t)u(t-\tau_l), \quad t \geqslant 0, \tag{1}$$

$$u(t) = \phi(t), \quad t \leqslant 0, \tag{2}$$

where the quantities τ_l are positive constants such that $0 < \tau_1 < \tau_2 < \cdots < \tau_m < +\infty$. We assume that the coefficients $a(t), b_l(t)$ are sufficiently smooth complex-valued functions such that $\mathrm{Re}(a) < 0$, $\sum_{l=1}^{m} |b_l| < -\mathrm{Re}(a)$. So for the Eqs. (1) and (2) there is a unique complex-valued solution $u \in H^1([0, +\infty))$ [11].

In 1989, Aziz and Monk solved the heat equation with continuous finite element method [1]. A kind of superconvergence $O(h^{2p+2})$ at the node points between time intervals was shown in [1] as well as in [6]. The basics of the cG

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methods as well as some references are also given in the textbook by Eriksson et al. [5]. In 2000, Chen proposed a new idea on superconvergence research in finite elements [3]. Later Chen put the idea into structure theory of superconvergence of finite elements [4]. The idea is to add some lower degree terms in the remainder of orthogonal expansion in an element so that the remainder satisfies more conditions in the element, and get a desired superclose function u_I to finite element solution u_h. In 2001, by use of this new idea Pan and Chen derived several new superconvergence results for the initial value problem of ordinary differential equation [9]. Based on above search we shall study superconvergence of continuous Galerkin finite element for the delay-differential Eqs. (1) and (2) with several terms.

For our element orthogonal analysis, we introduce Legendre's polynomials [4] in interval $E=[-1,1]$

$$l_0=1, l_1=s, l_2=\frac{1}{2}(3s^2-1), l_3=\frac{1}{2}(5s^3-3s), \ldots, l_n=\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}(s^2-1)^n, \ldots, \tag{3}$$

where the inner product $(l_i, l_j)=0$ if $i \neq j$, otherwise $(l_i, l_j)=2/(2j+1)$, $l(\pm 1)=(\pm 1)^j$. $l_n(s)$ has n distinct roots (n order Gauss points) in $(-1, 1)$. Integrating l_n, we get another family of polynomials [4]

$$M_0=1, M_1=s, M_2=\frac{1}{2}(s^2-1), M_3=\frac{1}{2}(s^3-s), \ldots, M_{n+1}=\frac{1}{2^n n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}}(s^2-1)^n, \ldots, \tag{4}$$

which has the following quasiorthogonal property: $(M_i, M_j) \neq 0$ if $i-j=0$ or $=\pm 2$, otherwise $(M_i, M_j)=0$. Obviously, $M_j(\pm 1)=0$ for $j \geqslant 2$. $M_{n+1}(s)$ has $n+1$ distinct roots (($n+1$) order Lobatto points: $-1=z_1<z_2<\cdots<z_{n+1}=1$ in E. Here and below, denote Sobolev space and its norm by $W^{k,p}(\cdot)$ and $\|u\|_{k,p,\cdot}$, respectively. If $p=2$, simply use $H^k(\cdot)$ and $\|u\|_k$.

In this paper we shall assume that the exact solution u is sufficient smooth for our purpose.

For $T>\tau_m$, the interval $J=[0, T]$ is partitioned uniformly into N elements. Let $h=T/2N \leqslant \tau_1$ denote half step-size of this partition and $J_n=[t_{n-1}, t_n]$ an element where $t_n=2nh$. Denote element midpoint by $t_{n-1/2}=(t_{n-1}+t_n)/2$ and this partition by

$$\mathscr{J}^h=\{J_n | n=1,2,\ldots,N\}.$$

Define p-degree finite element space to be

$$S^h=\{v \in C(J), v|_{J_n} \in \mathbf{P}_p, n=1,2,\ldots,N,\},$$

where $\mathbf{P}_p(J_n)$ denotes the space of all polynomials of degree $\leqslant p$ in J_n. Finally denote set of $(n+1)$-order Lobatto points in all elements in partition $\mathscr{J}^h$ by

$$Z_0=\{t_{ji}=t_{j-1/2}+h_j z_i, j=1,2,\ldots,N, i=1,2,\ldots,n+1\}.$$

For the sake of argument, let $\tau_0=0$, $\tau_{m+1}=+\infty$, $b_0(t)=a(t)$, $b_{m+1}(t)=0$, then Eq. (1) can be rewritten as

$$u'(t)=\sum_{l=0}^{m+1} b_l(t)u(t-\tau_l), \quad t \geqslant 0. \tag{5}$$

For some integer ρ, $0 \leqslant \rho \leqslant m$, $\tau_\rho \leqslant t < \tau_{\rho+1}$, in terms of initial condition (2), (5) yields

$$u'(t)-\sum_{l=0}^{\rho} b_l(t)u(t-\tau_l)=\sum_{l=\rho+1}^{m+1} b_l(t)\phi(t-\tau_l). \tag{6}$$

Multiplying by $v(t) \in L^2(J_n)=\{v | \int_{J_n} |v|^2\,\mathrm{d}t<\infty\}$ and integrating over the element J_n, we get

$$\int_{J_n}\left[u'(t)-\sum_{l=0}^{\rho} b_l(t)u(t-\tau_l)\right]v(t)\,\mathrm{d}t=\int_{J_n}\left[\sum_{l=\rho+1}^{m+1} b_l(t)\phi(t-\tau_l)\right]v(t)\,\mathrm{d}t, \quad \forall v(t) \in L^2(J_n). \tag{7}$$