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Fourier spectral approximation to long-time behaviour of the derivative three-dimensional Ginzburg–Landau equation☆

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Abstract

In this paper, we consider a derivative Ginzburg–Landau equation with periodic initial-value condition in three-dimensional space. A fully discrete Galerkin–Fourier spectral approximation scheme is constructed, and then the dynamical behaviour of the discrete system is analysed. Firstly, the existence of global attractors $\mathscr{A}_N^\tau$ of the discrete system are proved by a priori estimate of the discrete solution. Next, the convergence of approximate attractors is proved by error estimates of the discrete solution. Furthermore, the long-time convergence as $N \to \infty$ and $\tau \to 0$ simultaneously as well as the numerical long-time stability of the discrete scheme are obtained.

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1. Introduction and construction of discrete scheme

The study of the long-time behaviour of the solutions of dissipative partial differential equations is a major problem of mathematical physics, directly related to the understanding of turbulence. In general, the long-time behaviour of systems can be characterized by the existence and structure of global attractors. Nevertheless, further study for attractors depends on numerical experimentation results to a great extent, and then it is worth studying whether numerical results are reliable and computational schemes are suitable. This work began in the late 1980s. Our aim in this paper is to consider the discrete approximation scheme for the derivative Ginzburg–Landau equation in three space dimensions and to discuss related discrete dynamical issues.

The Ginzburg–Landau-type equation is an important nonlinear evolution equation and has many forms. In 1989, Brand and Deissler [1] derived firstly the one-dimensional derivative Ginzburg–Landau equation as follows:

$$u_t + \nu u_x = \chi u + (\gamma_r + \mathrm{i}\gamma_i)u_{xx} - (\beta_r + \mathrm{i}\beta_i)|u|^2 u - (\delta_r + \mathrm{i}\delta_i)|u|^4 u - (\lambda_r + \mathrm{i}\lambda_i)|u|^2 u_x - (\mu_r + \mathrm{i}\mu_i)u^2\overline{u}_x,$$

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where the overline denotes the complex conjugate. After this, there has been some work on the following more general derivative one-dimensional or two-dimensional Ginzburg–Landau equation derived by Doelman [3]:

$$u_t = \alpha_0 u + \alpha_1 u_{xx} + \alpha_2 |u|^2 u_x + \alpha_3 u^2 \overline{u}_x + \alpha_4 |u|^2 u + \alpha_5 |u|^{2\sigma} u \tag{*}$$

and

$$u_t = \rho u + (1+\mathrm{i}\gamma)\Delta u - (1+\mathrm{i}\mu)|u|^{2\sigma} u + \alpha(\lambda_1 \cdot \nabla \overline{u})u^2 + \beta(\lambda_2 \cdot \nabla u)|u|^2. \tag{**}$$

For example, Duan et al. [4,5] obtained the existence and uniqueness of global solutions for equation $(*)$ for the case of $\sigma = 2$; Guo and Gao [8] discussed the finite-dimensional behaviour of $(*)$ and proved the existence of a global attractor with finite Hausdorff and fractal dimensions; [6,10] obtained the existence and uniqueness of global solutions for the Cauchy problem of equation $(**)$ in two-dimensional spaces; Gao and Lin [7] Guo and Wang [9] proved the existence of a global attractor with finite Hausdorff and fractal dimensions for equation $(**)$ in two-dimensional spaces under appropriate assumptions on σ. However, relevant results are seldom for the case of three-dimensional space. The main reason is that some of the Sobolev interpolation inequalities used in one- or two-dimensional cases fail in the three-dimensional case. Especially, there is scarcely discussion about discrete dynamical issues for this kind of equations in existing literature.

Now we consider the periodic initial-value problem of the following three-dimensional derivative Ginzburg–Landau equation:

$$u_t - (\lambda + \mathrm{i}\alpha)\Delta u + (\kappa + \mathrm{i}\beta)|u|^{2\sigma} u - \gamma u + \lambda_1 \cdot |u|^2 \nabla u + \lambda_2 \cdot u^2 \nabla \overline{u} = 0, \quad x \in \mathbf{R^3},\ t \in \mathbf{R}^+, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R^3}, \tag{1.2}$$

$$u(x + L\mathbf{e}_j, t) = u(x,t), \quad x \in \mathbf{R^3},\ t \in \mathbf{R}^+, \tag{1.3}$$

where $\lambda, \alpha, \beta, \gamma, \kappa, \sigma$ are real constants and $\lambda > 0$, $\kappa > 0$, $\sigma > 0$, $\gamma > 0$. In our previous work [11], the existence and uniqueness of a global solution, the existence of global attractors and the estimates of upper bounds of their Hausdorff and fractal dimensions were proved for problem (1.1)–(1.3) under the following assumption on κ, β and σ:

$$\text{(H)} \qquad \frac{|\beta|}{\kappa} < \frac{\sqrt{2\sigma+1}}{\sigma} \quad \text{and} \quad \sigma > 2.$$

In this paper, our main intention is to study the discrete dynamical behaviour for (1.1)–(1.3). In what follows, we construct firstly a fully discrete Fourier spectral approximation scheme (1.4), (1.5). Then the existence of approximation attractors $\mathscr{A}_N^\tau$ is obtained in terms of t-independent a priori estimates of discrete solutions in Section 2 (Theorem 2.2). In addition, the convergence of approximate attractors as $N \to +\infty$ and $\tau \to 0$ is proved by the error estimates on $(0, +\infty)$ of the discrete solutions in Section 3 (Theorem 3.3). Next, the numerical stability of discrete scheme on $[0, T]$ is proved in Section 4 (Theorem 4.1). Finally, the long-time convergence as $N \to \infty$ and $\tau \to 0$ as well as the long-time numerical stability of the discrete scheme are obtained under the assumption (H) in Section 5 (Theorem 5.1 and Theorem 5.2).

For any given positive integer N, let $S_N = \mathrm{Span}\{(1/\sqrt{L^3})\mathrm{e}^{\mathrm{i}(2\pi/L)k\cdot x} : |k| \leqslant N\}$ and denote by $P_N : L_p^2(\Omega) \to S_N$ the orthogonal projection operator(see [2]).

Let τ be the discrete step in the variable t, $t_k = k\tau$, $u^k = u(x, t_k)$, $\overline{\partial}_t u^k = (1/\tau)(u^k - u^{k-1})$. We construct the Fourier spectral approximation scheme for problem (1.1)–(1.3) as follows: to find $u_N^k \in S_N$ such that

$$\begin{aligned}&(\overline{\partial}_t u_N^k - (\lambda + \mathrm{i}\alpha)\Delta u_N^k + (\kappa + \mathrm{i}\beta)|u_N^k|^{2\sigma} u_N^k - \gamma u_N^k, \varphi) \\ &\quad = -(\lambda_1 \cdot |u_N^k|^2 \nabla u_N^k + \lambda_2 \cdot (u_N^k)^2 \nabla \overline{u}_N^k, \varphi) \quad \forall \varphi \in S_N, k \geqslant 1\end{aligned} \tag{1.4}$$

and

$$u_N^0 = P_N u_0. \tag{1.5}$$