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Analysis of trigonometric implicit Runge–Kutta methods

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Abstract

Using generalized collocation techniques based on fitting functions that are trigonometric (rather than algebraic as in classical integrators), we develop a new class of multistage, one-step, variable stepsize, and variable coefficients implicit Runge–Kutta methods to solve oscillatory ODE problems. The coefficients of the methods are functions of the frequency and the stepsize. We refer to this class as trigonometric implicit Runge–Kutta (TIRK) methods. They integrate an equation exactly if its solution is a trigonometric polynomial with a known frequency. We characterize the order and A-stability of the methods and establish results similar to that of classical algebraic collocation RK methods.

Keywords: Oscillatory ODEs; Variable stepsize; Variable coefficients; Trigonometric collocation; Implicit RK

1. Introduction

ODE problems that are intermittently oscillatory and/or stiff represent an important class of problems that arise in practice. In biology for example, most living organisms experience circadian (i.e., daily) oscillations that can be represented by periodic kinetics ODE models of period 24 h. Another noteworthy example is the ruby laser oscillator [2] which is difficult to solve because it is initially stiff, then mildly damped before becoming highly oscillatory, and eventually approaching a constant steady state. A number of numerical methods for oscillatory problems have been developed. Many of these methods (e.g., Runge–Kutta–Nyström) are quite generic. Some of them can however take advantage of special properties of the solution that may be known in advance. What is more rare (and harder to implement) are type-insensitive integrators that can adaptively switch between different modes of the solution (stiff, non-stiff, oscillatory).

For a periodic ODE problem whose frequency, or a reasonable estimate of it, is known in advance, it can be advantageous to tune a method to take this estimate as a prior knowledge. We describe a new class of such methods. We use generalized collocation techniques based on fitting functions that are trigonometric (rather than algebraic as in classical integrators) to develop a class of multistage, one-step, variable stepsize, and variable coefficients implicit Runge–Kutta methods to solve oscillatory ODE problems. The coefficients of the methods are functions of the frequency and the stepsize. We refer to this class as trigonometric implicit Runge–Kutta (TIRK) methods. We give the general

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forms of their coefficients and their stability functions. Although there have been similar methods constructed through the usage of a trigonometric basis, these general results are new. Moreover, we show the existence of A-stable methods in the families of the two-, three- and four-stages methods with symmetric collocation points.

The organization of the paper is as follows. We start by briefly recalling Ozawa's results in Section 2. We then define and motivate TIRK methods in Section 3. We subsequently characterize their coefficients in Section 4 and reveal their stability functions in Section 5. Finally, in Section 6 we present some numerical results and then give some concluding remarks in Section 7.

2. Functionally fitted RK methods

Classical RK methods are polynomially fitted in the sense that they integrate any ODE problem exactly if its solution is a polynomial up to some degree, though the methods have an error for general ODEs. Likewise, other RK methods relying on exponentials exist [8,9,23,24]. Extending further, Ozawa [17–19] studied a more general family of *functionally fitted* RK methods that are constructed by first choosing a set of scalar *basis functions* $\{u_k(t)\}_{k=1}^s$ that are linearly independent and sufficiently smooth, and then generating an s-stage RK method that will exactly integrate any ODE problem whose solution can be expressed as a linear combination of the basis functions. Ozawa investigated the existence of these methods. We briefly summarize his main results here since our new results build on them.

Consider the problem of solving a system of first order differential equations of dimension d

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(t)), \quad \boldsymbol{y}: \mathbb{R} \to \mathbb{R}^d, \ \boldsymbol{f}: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \ t \in [0, T] \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \quad \text{initial condition.} \end{cases} \tag{1}$$

A given s-stage RK-method is defined by its Butcher-tableau

$$\begin{array}{c|c} \boldsymbol{c} & \boldsymbol{A} \\ \hline & \boldsymbol{b}^{\mathrm{T}} \end{array}, \quad \boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{s \times s}, \quad \boldsymbol{b} \in \mathbb{R}^s, \ \boldsymbol{c} = \boldsymbol{A}\boldsymbol{e}, \ \boldsymbol{e} = (1, \ldots, 1)^{\mathrm{T}},$$

in which, for an explicit RK-method, $\boldsymbol{A}$ is strictly lower triangular and $c_1 = 0$. Using the current value $\boldsymbol{y}_n$ at t_n and taking an appropriate stepsize h, the next iterate of the integration scheme is computed as

$$\begin{cases} \boldsymbol{Y}_i = \boldsymbol{y}_n + h\sum_{j=1}^s a_{ij}\boldsymbol{f}(t_n + c_j h, \boldsymbol{Y}_j), \quad i = 1, \ldots, s, \\ \boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\sum_{j=1}^s b_j \boldsymbol{f}(t_n + c_j h, \boldsymbol{Y}_j). \end{cases}$$

These relations are often represented compactly using a Kronecker tensor product notation. In the scalar case (i.e., $d = 1$), this becomes:

$$\boldsymbol{Y} = \boldsymbol{e} y_n + h\boldsymbol{A} f(\boldsymbol{e}t_n + \boldsymbol{c}h, \boldsymbol{Y}) \in \mathbb{R}^s,$$

$$y_{n+1} = y_n + h\boldsymbol{b}^{\mathrm{T}} f(\boldsymbol{e}t_n + \boldsymbol{c}h, \boldsymbol{Y}) \in \mathbb{R},$$

where $\boldsymbol{Y} = (Y_1, \ldots, Y_s)^{\mathrm{T}}$ and $f(\boldsymbol{e}t_n + \boldsymbol{c}h, \boldsymbol{Y}) = f(t_n + c_1 h, Y_1), \ldots, f(t_n + c_s h, Y_s))^{\mathrm{T}}$. Functionally fitted RK methods are constructed by demanding that the given scalar basis functions $\{u_k(t)\}_{k=1}^s$ satisfy the integration exactly.

Definition 1 (*Functionally fitted RK*). An s-stage RK method is a functionally fitted RK (or a generalized collocation RK) method with respect to the basis functions $\{u_k(t)\}_{k=1}^s$ if each u_k satisfies the following relations:

$$u_k(\boldsymbol{e}t + \boldsymbol{c}h) = \boldsymbol{e}u_k(t) + h\boldsymbol{A}(t, h)u_k'(\boldsymbol{e}t + \boldsymbol{c}h),$$

$$u_k(t + h) = u_k(t) + h\boldsymbol{b}(t, h)^{\mathrm{T}}u_k'(\boldsymbol{e}t + \boldsymbol{c}h). \tag{2}$$

Writing these equalities in matrix form, we get a linear system that can be solved for $\boldsymbol{A}$ and $\boldsymbol{b}$, yielding a method with variable coefficients that generally depend on t and h. The parameters $(c_i)_{i=1}^s$ are usually taken in [0, 1] and we assume that they are distinct. Ozawa [18] showed that $\boldsymbol{A}$ and $\boldsymbol{b}$ are uniquely determined for small $h > 0$ and $t \in [0, T]$ if the Wronskian $W(u_1^{(1)}, \ldots, u_s^{(1)})(t) \neq 0$. Other weaker conditions were given in [16].