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On regular and singular interval systems

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Dedicated to the late Professor Teruo Sunaga, Fukuoka, Japan

Abstract

We give a survey on interval linear systems discussing problems for regular systems as well as for singular ones. We consider several solution sets and direct methods to enclose them. Moreover we study iterative methods, particularly the total step method as the basis for other ones. We also use this method for enclosing solutions of singular linear systems.

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1. Introduction

In this paper we consider the set of linear systems of equations $Ax = b$ where A is a matrix which varies in a given real $n \times n$ interval matrix $[A]$ and where b is any vector contained in a given real interval vector $[b]$ with n components. This set is usually called 'interval linear system of equations' or shortly 'interval linear system'. It is denoted by

$$[A]x = [b] \tag{1.1}$$

which is misleading in some sense since one normally does not look for a real vector x^* or an interval vector $[x]^*$ such that (1.1) holds for $x = x^*$ or $x = [x]^*$ algebraically using interval arithmetic. Nevertheless we will keep up the traditional formal notation (1.1) throughout the paper. One of the true problems in connection with (1.1) consists in enclosing (possibly tightly) the solution set

$$\Sigma = \Sigma_{\exists\exists} = \{x \mid (\exists A \in [A])(\exists b \in [b])(Ax = b)\} \tag{1.2}$$

by an interval vector $[x]^*$. This is usually called 'verification of the solutions $x \in \Sigma$' provided that $[A]$ is regular, i.e., it does not contain a singular matrix. A vector $[x]^* \supseteq \Sigma$ is sometimes called a 'solution of (1.1)'. For singular matrices $[A]$, i.e., matrices which contain at least one singular matrix A as element this terminology has to be modified since a compact interval vector can never contain a non-trivial affine subspace which occurs in consistent singular linear systems $Ax = b$. We discuss this problem in Section 5 where we also report on the convergence of the total step method

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if the spectral radius $\rho(|[A]|)$ of the absolute value $|[A]| \in \mathbb{R}^{n\times n}$ of $[A]$ no longer satisfies the usual convergence criterion $\rho(|[A]|) < 1$. In this way we generalize the set of admissible regular interval matrices $[A]$ for this method. Moreover, we allow systems with singular matrices A at the boundary of $[A]$. In Section 2 we report on possible origins of interval linear systems and on particular subsets of Σ from (1.2), in Section 3 we mention direct interval methods for enclosing a solution $[x]^*$ of (1.1) presenting a new unpublished necessary and sufficient criterion for the feasibility of the interval Gaussian algorithm (cf. Theorem 3.1c). Section 4 is devoted to iterative methods with emphasis on the total step method for interval data. We illustrate the central role of this simple method by deriving connections to methods in relevant interval software. Nearly all of our results are not yet contained in textbooks like [1, 30]. It was our aim to give a survey on them without being exhaustive. Due to page limit we had to omit such important topics as complexity, sparsity and subspace methods. Moreover, our citations had to be very restrictive. They should be understood as starting point but by no means as a complete list of references in this area.

By $\mathbb{IR}$, $\mathbb{IR}^n$, $\mathbb{IR}^{n\times n}$ we denote the set of intervals, the set of interval vectors with n components and the set of $n \times n$ interval matrices, respectively. By 'interval' we mean here a real compact interval. As already indicated we write interval quantities in brackets with the exception of point quantities (i.e., degenerate interval quantities) which we identify with the element which they contain. Examples are the identity matrix I and the vector $e=(1, 1, \ldots, 1)^{\mathrm{T}}$. We use the notation $[A] = [\underline{A}, \overline{A}] = ([a]_{ij}) = ([\underline{a}_{ij}, \overline{a}_{ij}]) \in \mathbb{IR}^{n\times n}$ simultaneously without further reference, and we proceed similarly for the elements of $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$, $\mathbb{IR}$ and $\mathbb{IR}^n$. For an interval $[a]$ we introduce the midpoint $\check{a} = (\underline{a} + \overline{a})/2$, the absolute value $|[a]| = \max\{|\underline{a}|, |\overline{a}|\}$, the radius $\mathrm{rad}([a]) = (\overline{a} - \underline{a})/2$ and the interior $\mathrm{int}([a]) = (\underline{a}, \overline{a})$. For interval vectors and interval matrices these quantities are defined entrywise, for instance $|[A]| = (|[a]_{ij}|) \in \mathbb{R}^{n\times n}$. For their properties and for the interval arithmetic on which most of our results are based we refer to the introductory chapters of the textbooks which we just mentioned—see also the pioneering work of [40].

As usual we call a vector $x \in \mathbb{R}^n$ non-negative if $x_i \geqslant 0$ for $i = 1, \ldots, n$, writing $x \geqslant 0$ in this case. By $x > 0$ we denote a vector whose entries all are positive. For matrices we apply this definition analogously. If $A \in \mathbb{R}^{n\times n}$ only has non-positive off-diagonal entries and if it has a non-negative inverse then A is called an M matrix. An interval matrix $[A] \in \mathbb{IR}^{n\times n}$ is called an M matrix if each $A \in [A]$ has this property. It is an H matrix if its comparison matrix $\langle [A] \rangle = (c_{ij}) \in \mathbb{R}^{n\times n}$ is an M matrix where the entries of a comparison matrix are defined by $c_{ij} = \min\{|a| \mid a \in [a]_{ij}\}$ if $i = j$ and $c_{ij} = -|[a]_{ij}|$ if $i \neq j$.

2. Solution sets

There are several mechanisms which may lead to *interval* linear systems. Common to all is the aim to enclose the input data of one or several point linear systems $Ax = b$. The need for this action can occur during a computational process or is already immanent in the underlying mathematical problem by virtue of inexact input data. Examples for the first category are conversion errors like the conversion from decimal to binary system, and rounding errors when computing for instance the entries of A and b. Thus the decimal number 0.1 cannot be represented with finitely many digits as a binary number but is a periodic dual fraction. Therefore input data which are exactly represented in the decimal floating point system may fail to be represented exactly as binary floating point numbers. If one wants to compute with the exact input data nevertheless one has to compute with machine representable bounds for these input data. This directly leads to interval linear systems. Analogously one obtains such systems when dealing with rounding errors. Two examples for the second category are non-linear systems of equations with Newton's method hidden behind and Leontief's static open input–output model. We will shortly describe the situation in both cases.

Example 2.1. Look for a zero x^* of a given continuously differentiable function $f = (f_i) : x \in \mathbb{R}^n \to \mathbb{R}^n$ and assume that you are given an approximation $\tilde{x}$ for x^*. Assume that $[x] \in \mathbb{IR}^n$ is known such that $\tilde{x}, x^* \in [x]$. Using Taylor expansion one obtains

$$\begin{aligned} 0 = f(x^*) &= f(\tilde{x}) + \int_0^1 f'(\tilde{x} + t(x^* - \tilde{x}))\,\mathrm{d}t \cdot (x^* - \tilde{x}) \\ &= f(\tilde{x}) + (\mathrm{grad}\, f_i(\xi_i))^{\mathrm{T}} \cdot (x^* - \tilde{x}) \in f(\tilde{x}) + f'([x])([x] - \tilde{x}), \end{aligned}$$