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Remarks on a posteriori error estimation for finite element solutions

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Abstract

We utilize the classical hypercircle method and the lowest-order Raviart–Thomas $H(\mathrm{div})$ element to obtain a posteriori error estimates of the P_1 finite element solutions for 2D Poisson's equation. A few other estimation methods are also discussed for comparison. We give some theoretical and numerical results to see the effectiveness of the methods.

Keywords: Hypercircle method; Error bound; P_1 triangular element; Raviart–Thomas triangular element

1. Introduction

The finite element method is now used as a representative numerical method for partial differential equations. Mathematical analysis of such a method have been also extensively performed, and the so-called "a priori" error estimation is now popular [3–5,7,8]. Moreover, "a posteriori" error estimation has also become available utilizing some information of the obtained finite element solutions, and can be used as a basis of adaptive computation [1,3,7,8,10,12,13]. In this paper, we will present some results on a special a posteriori estimation method.

As a model problem, we consider the 2D Poisson equation with the homogeneous Dirichlet boundary condition: Given f, find u that satisfies

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where Ω is a bounded polygonal domain with boundary $\partial\Omega$, f a given function defined on Ω, and u an unknown function in Ω. In the finite element method (FEM), we usually use the following weak formulation of the above model problem: Given $f \in L_2(\Omega)$, find $u \in U := H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v); \quad \forall v \in U, \tag{2}$$

where $(\cdot,\cdot)$ denotes the inner product of $L_2(\Omega)$ or $L_2(\Omega)^2$. Moreover, $L_2(\Omega)$ and $H_0^1(\Omega)$ are usual Sobolev spaces associated to Ω [5].

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To solve the above problem by a typical P_1 FEM, we first consider a regular family of triangulations $\{\mathscr{T}^h\}_{h>0}$ of Ω, and then construct the P_1 (i.e., piecewise linear) finite element space U^h as a subspace of $U = H_0^1(\Omega)$ for each $\mathscr{T}^h$. Usually, h denotes the maximum edge length of all triangles in the triangulation. Finally, the finite element solution $u_h \in U^h$ is specified by

$$(\nabla u_h, \nabla v_h) = (f, v_h); \quad \forall v_h \in U^h. \tag{3}$$

For the present u_h, we can obtain the following well-known a priori estimates:

$$\|\nabla u - \nabla u_h\| \leqslant C_1 h^\delta \|u\|_{H^{1+\delta}(\Omega)} \leqslant C_2 h^\delta \|f\|, \quad \|u - u_h\| \leqslant C_2^2 h^{2\delta} \|f\|, \tag{4}$$

where $\|\cdot\|$ denotes the norm of $L_2(\Omega)$ or $L_2(\Omega)^2$, C_1 and C_2 are positive constants dependent on Ω and the family of triangulations only, δ is a constant such that $\frac{1}{2} < \delta \leqslant 1$ depending only on the maximum interior angle of Ω, and $H^{1+\delta}(\Omega)$ is the (fractional) Sobolev space. In particular, $\delta = 1$ when Ω is a convex polygonal domain. In this type of a priori estimation, the approximate solution u_h does not appear in the right-hand sides of the inequalities. Instead, some informations on u and/or f are used. Furthermore, we can also obtain similar a priori error estimates in some other norms. For quantitative purposes, the positive constants like C_1 and C_2 above should be evaluated beforehand, although such evaluation is not necessarily easy.

Another error estimation method developing rapidly is the so-called a posteriori method, where the approximate solution u_h is also used in the right-hand sides. Such a method is also used as basis of adaptive computation. Various methods have been developed in this category, and one of the most classical one is that based on the hypercircle method [11], which does not require any positive constants like C_1, C_2 for estimation in some special norms. However, it has been almost forgotten for a long time: in fact, its implementation in FEM is not easy from strict viewpoint. However, in some very special problems, we can apply such an idea after slightly relaxing the severe conditions required in the original hypercircle method. Such an approach was proposed by Destuynder and Métivet [6] utilizing the Raviart–Thomas $H(\text{div})$-triangular element and the mixed FEM [4]. See also [2,9] for related works.

In this paper, we will present some theoretical results on such an approach together with related methods and numerical results.

2. Hypercircle method

Let us explain the essence of the hypercircle method for the model problem (1), which is, for a given $f \in L_2(\Omega)$, to find $u \in U = H_0^1(\Omega)$ such that $-\triangle u = f$. The Poisson differential equation can be decomposed into

$$p = \nabla u, \quad \operatorname{div} p + f = 0. \tag{5}$$

Thus we naturally introduce the following affine set for the given $f \in L_2(\Omega)$:

$$H_f(\text{div}; \Omega) := \{q \in L_2(\Omega)^2; \operatorname{div} q + f = 0\} \subset H(\text{div}; \Omega) := \{q \in L_2(\Omega)^2; \operatorname{div} q \in L_2(\Omega)\}. \tag{6}$$

Clearly, $p = \nabla u$ belongs to this set, and we can easily obtain the Prager–Synge identity [11]:

$$\|\nabla u - \nabla v\|^2 + \|p - q\|^2 = \|\nabla v - q\|^2; \quad \forall v \in H_0^1(\Omega), \ \forall q \in H_f(\text{div}; \Omega). \tag{7}$$

Essentially, this is the Pythagorean theorem based on the orthogonality condition $\nabla u - \nabla v \perp p - q$ in $L_2(\Omega)^2$, where the vertex of right angle is at $\nabla u = p$. Thus the three points $\nabla u = p$, ∇v and q lie on a hypercircle whose center and radius are, respectively, $\frac{\nabla v + q}{2}$ and $\|\nabla u - \frac{\nabla v + q}{2}\| = \frac{1}{2}\|\nabla v - q\|$.

The idea of the hypercircle method is very simple. If we take v as the finite element solution u_h, which is surely in U since $U^h \subset U$, we have the estimates

$$\|\nabla u - \nabla u_h\| \leqslant \|\nabla u_h - q\|, \quad \|\nabla u - q\| \leqslant \|\nabla u_h - q\|, \quad \left\|\nabla u - \frac{\nabla u_h + q}{2}\right\| = \frac{1}{2}\|\nabla u_h - q\|, \tag{8}$$