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Existence of positive entire solutions for singular and non-singular quasi-linear elliptic equation☆

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Abstract

We consider the quasilinear elliptic equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda m(x)u^{\gamma} = 0$ in domain $\Omega = \mathbf{R}^{\mathbf{N}}$, where $\lambda > 0$, $p > 1$. Under several hypotheses on the $m(x)$, γ, we obtain the existence of positive entire solutions.

MSC: 35J05; 35J65; 35K05; 35K60

Keywords: Quasi-linear elliptic equation; Positive entire solution; Existence

1. Introduction

In this paper, we are concerned with the existence of positive entire solutions of quasilinear elliptic equations of the type

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda m(x)u^{\gamma} = 0, \quad x \in \Omega = \mathbf{R}^{\mathbf{N}}, \tag{1.1}$$

where $m(x) = m(|x|) \in C(\mathbf{R}^{+})$, $\lambda > 0$, $p > 1$, and $0 \geqslant \gamma \geqslant -(p-1)$ or $\gamma > 0$. By a positive entire solution of Eq. (1.1), we mean a positive function $u \in C^1(\mathbf{R}^{\mathbf{N}})$ which satisfies (1.1) at every point of $\mathbf{R}^{\mathbf{N}}$(see [10] and references therein). If $\lim_{r\to\infty} u(r) = 0$, we call it a positive decaying solution.

Equations of the above form are mathematical models occurring in studies of the p-Laplace equation, generalized reaction–diffusion theory [19], non-Newtonian fluid theory [2,25], non-Newtonian filtration [18] and the turbulent flow of a gas in porous medium [8]. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

In recent years, the existence and uniqueness of the positive solutions for the quasilinear eigenvalue problems

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda f(u) = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$u(x) = 0 \quad \partial\Omega, \tag{1.3}$$

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with $\lambda > 0$, $p > 1$, $\Omega \subset \mathbf{R^N}$, $N \geqslant 2$ have been studied by many authors, see [9–17,27,28,30,32,34] and the references therein. When f is strictly increasing on $\mathbf{R}^+$, $f(0) = 0$, $\lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leqslant \alpha_1 + \alpha_2 s^{\mu}$, $0 < \mu < p - 1$, $\alpha_1, \alpha_2 > 0$, it was shown in [11] that there exist at least two positive solutions for Eqs. (1.2)–(1.3) when λ is sufficiently large. If $\lim_{s\to 0^+} \inf f(s)/s^{p-1} > 0$, $f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$, it was proved in [15] that problems (1.2)–(1.3) has a unique positive solution when λ is sufficiently large. Moreover, it was also shown in [14] that problems (1.2)–(1.3) have a unique positive large solution and at least one positive small solution when λ is large if f is non-decreasing; there exist $\alpha_1, \alpha_2 > 0$ such that $f(s) \leqslant \alpha_1 + \alpha_2 s^{\beta}$, $0 < \beta < p - 1$; $\lim_{s\to 0^+} f(s)/(s^{p-1}) = 0$, and there exist $T, Y > 0$ with $Y \geqslant T$ such that

$$(f(s)/s^{p-1})' > 0 \quad \text{for } s \in (0, T)$$

and

$$(f(s)/s^{p-1})' < 0 \quad \text{for } s > Y.$$

Recently, Hai [17] considered the case when Ω is an annular domain, and obtained the existence of positive large solutions for problems (1.2)–(1.3) when λ is sufficiently small. Xuan & Chen proved in [3] the singular problem (1.1), (1.3) has a unique positive radial solution if m is a continuous function and positive on $\overline{\Omega} = B_R$ (here B_R is a ball). In contrast to these cases, it seems that very little is known about the existence of entire solutions for (1.1) with singular and non-singular cases except for the work by [10,33,31]. In this paper, we show the existence of positive entire solution to singular problem (1.1) vanishing at infinity under relaxed decay and positivity conditions on the function $m(x)$ with other new condition, our results extend part works by [35,21,7,20], and complement part works by [10,33,31,3]. For $p = 2$, the related results to a singular semilinear elliptic the boundary value problem,

$$\begin{cases} \triangle u + \lambda m(x) u^{\gamma} = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

have been extensively studied when $\Omega \subset \mathbf{R^N}$, see [4–7,20,21,23,24,35]. When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be lost or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ can be founded in [11,15]. In the last section, we study existence of positive radial entire solutions for non-singular quasilinear elliptic equation, this results extend and complement previous part works by Lair–Shaker [22].

2. Singular case

In this section, we discuss the existence of positive radial entire solutions of problem (1.1) with $\Omega = \mathbf{R^N}$, $N \geqslant 2$. In this case, Eq. (1.1) is equivalent to

$$(r^{N-1}|u'|^{p-2}u')' + \lambda r^{N-1} m(r) u^{\gamma} = 0, \quad r > 0, \quad \lambda > 0, \tag{2.1}$$

where $r = |x|$. If $u = u(r)$ is a solution of (2.1) with $u(0) = \mu$, then u satisfies $u'(0) = 0$. Therefore, Eq. (2.1) is equivalent to

$$u(r) = \mu - \int_0^r \Phi_p^{-1}\left[\lambda s^{1-N} \int_0^s t^{N-1} m(t) u^{\gamma}(t)\,\mathrm{d}t\right] \mathrm{d}s \quad \text{for } r \geqslant 0 \tag{2.2}$$

and if $u = u(r)$ is a positive decaying solution of (2.1), then u satisfies

$$u(r) = \int_r^{\infty} \Phi_p^{-1}\left[\lambda s^{1-N} \int_0^s t^{N-1} m(t) u^{\gamma}(t)\,\mathrm{d}t\right] \mathrm{d}s \quad \text{for } r \geqslant 0, \tag{2.3}$$

where

$$\Phi_p^{-1}(u) = \begin{cases} u^{1/(p-1)} & \text{if } u \geqslant 0, \\ -(-u)^{1/(p-1)} & \text{if } u < 0. \end{cases}$$