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Artificial boundary conditions for parabolic Volterra integro-differential equations on unbounded two-dimensional domains

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Abstract

In this paper we study the numerical solution of parabolic Volterra integro-differential equations on certain unbounded two-dimensional spatial domains. The method is based on the introduction of a feasible artificial boundary and the derivation of corresponding artificial (fully transparent) boundary conditions. Two examples illustrate the application and numerical performance of the method.

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Keywords: Partial Volterra integro-differential equation; Unbounded spatial domain; Artificial boundary conditions; Numerical solution

1. Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be a semi-infinite strip domain with boundary $\Gamma = \Gamma_i \cup \Gamma_U \cup \Gamma_L$ (as shown in Fig. 1). Γ_U and Γ_L are assumed to be parallel.

Consider the following initial-boundary-value problem for a parabolic equation with memory term

$$\frac{\partial u}{\partial t} + \int_0^t k(x, t-\tau)u(x,\tau)\,\mathrm{d}\tau = \nabla(\alpha(x)\nabla u) - \beta(x)u + f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

$$u = g(x,t), \quad (x,t) \in \Gamma \times (0,T], \tag{1.2}$$

$$u(x,0) = u_0(x) \quad x \in \Omega, \tag{1.3}$$

$$u(x,t) \to 0 \quad \text{as } x_1 \to +\infty. \tag{1.4}$$

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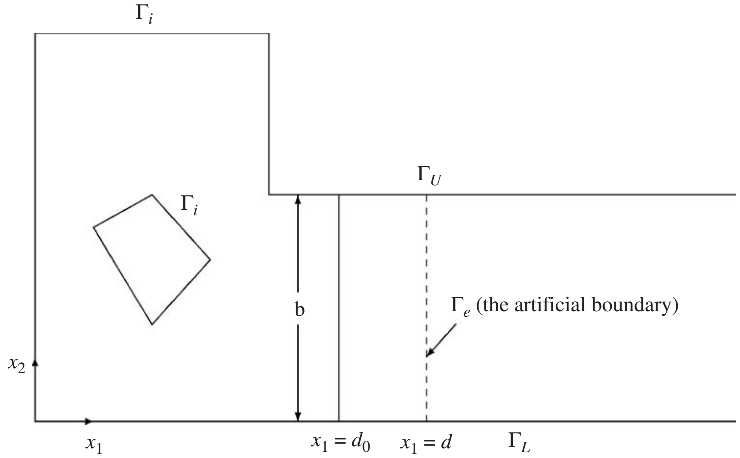

Fig. 1. Unbounded domain Ω and artificial boundary Γ_e.

We assume that:

(i) $\alpha(x) - 1 \geqslant 0$, $\beta(x) - \beta_0 \geqslant 0$ (β_0 is a non-negative constant), and $u_0(x)$ has compact support;
$\mathrm{Supp}\{\alpha(x) - 1\} \subset \bar{\Omega}_0 := \{x | x \in \bar{\Omega} \text{ and } x_1 \leqslant d_0\}$,

$\mathrm{Supp}\{\beta(x) - \beta_0\} \subset \bar{\Omega}_0$,

$\mathrm{Supp}\{u_0(x)\} \subset \bar{\Omega}_0$.

(ii) $f(x, t)$ and $g(x, t)$ have compact support:
$\mathrm{Supp}\{f\} \subset \bar{\Omega}_0 \times [0, T]$ and $\mathrm{Supp}\{g\} \subset \bar{\Omega}_0 \times [0, T]$.

(iii) $k(x, t) \equiv k_0(t)$ for $x_1 \geqslant d_0$.

In order to solve this problem numerically we introduce an artificial boundary $\Gamma_e \times [0, T]$ defined by

$$\Gamma_e := \{x = (x_1, x_2) \in \Omega : x_1 = d,\ 0 \leqslant x_2 \leqslant b,\ d \geqslant d_0\}.$$

This artificial boundary divides the domain $\bar{\Omega} \times [0, T]$ into two parts, the *bounded* part $\bar{\Omega}_i \times [0, T]$ and the *unbounded* part $\Omega_e \times [0, T]$

$$\Omega_i = \{x | x \in \Omega \text{ and } x_1 < d\}, \quad \Omega_e = \Omega \backslash \overline{\Omega}_i.$$

Our aim is to present a feasible and computationally effective numerical scheme for the approximate solution of the problem (1.1)–(1.4) on the bounded domain $\bar{\Omega}_i \times [0, T]$. This hinges on the derivation of a suitable artificial boundary condition on the given artificial boundary $\Gamma_e \times [0, T]$.

The artificial boundary method was introduced and analyzed for elliptic problems in [6,7]; see also [8,3]. In [4,5], these artificial boundary techniques were extended to the heat equation and related parabolic PDEs, and their approach was subsequently generalized [9] to one-dimensional “non-local” parabolic equations containing a memory term given by a (regular or weakly singular) Volterra integral operator.

The purpose of the present paper is to describe the computational form of the artificial boundary method for parabolic Volterra integro-differential equations of the form (1.1) on unbounded two-dimensional spatial domains given essentially by a semi-infinite strip, and to illustrate its numerical performance. It will be seen in Sections 2 and 3 that passing from one to two (or more) spatial dimensions is not trivial (compare also [7,8,4]).

The content of the paper is as follows. In Section 2 we derive the corresponding transparent artificial boundary condition on $\Gamma_e \times [0, T]$, significantly extending the approach in [9]. The reduction of the original problem (1.1)–(1.4) to the bounded domain $\Omega_i \times [0, T]$ is presented in Section 3. Here, we also state and prove a first result dealing with the