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Weak local linear discretizations for stochastic differential equations: Convergence and numerical schemes

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Abstract

Weak local linear (WLL) discretizations are playing an increasing role in the construction of effective numerical integrators and inference methods for stochastic differential equations (SDEs) with additive noise. However, due to limitations in the existing numerical implementations of WLL discretizations, the resulting integrators and inference methods have either been restricted to particular classes of autonomous SDEs or showed low computational efficiency. Another limitation is the absence of a systematic theoretical study of the rate of convergence of the WLL discretizations and numerical integratos. This task is the main purpose of the present paper. A second goal is introducing a new WLL scheme that overcomes the numerical limitations mentioned above. Additionally, a comparative analysis between the new WLL scheme and some conventional weak integrators is also presented.

Keywords: Local linearization; Weak schemes; Weak convergence; Stochastic differential equations

1. Introduction

In a number of problems in mathematical physics, biology, economics, finance and other fields the evaluation of Wiener functional space integrals and the estimation of diffusion processes play a prominent role. Examples are the computation of the Feynman–Kac formula in the analysis of wave scattering in random media [3], the computation of Lyapunov exponents of random dynamical systems [8], the estimation of the investment distribution that maximizes the expected utility function in an optimal portfolio problem [22], the estimation of continuous time stochastic volatilities models for stock prices [6], etc. In the solution of this kind of problems, weak numerical integrators for Stochastic Differential Equations (SDEs) have become an essential tool [31,32,15,19]. During the last years, much progress has been made in the study of weak numerical integrators (see [21] for an updated review of these methods). Well-known are, for instance, the Euler, the Milstein, the Talay–Tubaro extrapolation, the Runge–Kutta and the local linearization (LL) methods.

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This paper focus on the class of weak LL schemes. The main idea of the LL method is obtaining a discretization of the d-dimensional nonlinear SDE

$$\begin{aligned} d\mathbf{x}(t) &= \mathbf{f}(t,\mathbf{x}(t))\,dt + \mathbf{G}(t)\,d\mathbf{w}(t), \quad t \in [t_0, T], \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned} \tag{1}$$

by means of a local (piecewise) linear approximation of the drift coefficient $\mathbf{f}$. Here, $\mathbf{w}$ is a m-dimensional standard Wiener process and $\mathbf{G}$ is a $d \times m$ matrix function. Two different ways of linearization have been considered, namely, either based on truncated Taylor or Ito–Taylor expansions of the function $\mathbf{f}$ [18,2,24,13]. They lead to two different weak LL discretizations that can be expressed by the recurrent relation

$$\mathbf{y}_{t_{n+1}} = \mathbf{y}_{t_n} + \boldsymbol{\phi}_\beta(t_n, \mathbf{y}_{t_n}; t_{n+1} - t_n) + \boldsymbol{\eta}(t_n, \mathbf{y}_{t_n}; t_{n+1} - t_n), \tag{2}$$

evaluated at an increasing sequence of discrete times $t_n \in [t_0, T]$, where $\mathbf{y}_{t_0} = \mathbf{x}_0$,

$$\boldsymbol{\phi}_\beta(t, \mathbf{y}; \delta) = \int_0^\delta e^{(\partial\mathbf{f}(t,\mathbf{y})/\partial\mathbf{y})(\delta - s)}(\mathbf{f}(t,\mathbf{y}) + \mathbf{b}_\beta(t,\mathbf{y})s)\,ds, \tag{3}$$

with

$$\mathbf{b}_\beta(t,\mathbf{y}) = \left\{ \begin{array}{ll} \dfrac{\partial\mathbf{f}(t,\mathbf{y})}{\partial t}, & \beta = 1 \\ \dfrac{\partial\mathbf{f}(t,\mathbf{y})}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{k,l=1}^{d} (\mathbf{G}(t)\mathbf{G}^\top(t))^{k,l}\, \dfrac{\partial^2\mathbf{f}(t,\mathbf{y})}{\partial\mathbf{y}^k\partial\mathbf{y}^l}, & \beta = 2 \end{array} \right\},$$

for all $(t, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^d$ and $\delta > 0$; and $\boldsymbol{\eta}(t, \mathbf{y}; \delta)$ is a zero mean stochastic process with variance matrix

$$\boldsymbol{\Sigma}(t, \mathbf{y}; \delta) = \int_0^\delta e^{(\partial\mathbf{f}(t,\mathbf{y})/\partial\mathbf{y})(\delta - s)}\mathbf{G}(t+s)\mathbf{G}^\top(t+s)e^{(\partial\mathbf{f}(t,\mathbf{y})/\partial\mathbf{y})^\top(\delta - s)}\,ds. \tag{4}$$

The discretization with $\beta=1$ corresponds to the truncated Taylor expansion, while other one corresponds to the truncated Ito–Taylor expansion.

On the basis of these two discretizations, various weak LL schemes have been proposed [18,24,16], which differ in respect to the way of computing the integrals (3) and (4). In particular, the schemes introduced in [18,24] have played a prominent role in the construction of effective inference methods for SDEs [23,25,5,27], in the estimation of distribution functions in Monte Carlo Markov Chain methods [29,20,9] and the simulation of likelihood functions [17]. Moreover, the extensive simulation studies carried out in the papers just mentioned have showed that these LL schemes posses high numerical stability and remarkable computational efficiency. However, due to limitations in the numerical implementations of such LL schemes, they have been restricted to the class of autonomous SDEs with nonsingular Jacobian for the drift term. Moreover, the absence of a systematic theoretical study for these schemes has limited the study of the inference methods as well. Up to now, only the consistency of them has been studied in [28], while none estimate for the convergence rate has been given. On the other hand, an order two LL scheme recently introduced in [16] overcomes the restrictions of the previous schemes. However, this is based on quadrature formulas that, as will be shown in this paper, are inefficient and unnecessary.

The main purpose of this paper is studying the convergence rate of both, the LL discretizations (2) and the first two LL schemes mentioned above. A second goal is introducing a new weak LL scheme that overcomes the drawbacks of the preceding LL schemes. Additionally, a simulation study is presented in order to illustrate the performance of the proposed integrator, which includes a comparison with conventional weak integrators as well.

2. Notations and preliminaries results

Let $(\Omega, \mathscr{F}, \mathscr{P})$ be an underlying complete probability space and $\{\mathscr{F}_t, t \geqslant t_0\}$ be an increasing right continuous family of complete sub σ-algebras of $\mathscr{F}$ associated to the m-dimensional standard Wiener process $\mathbf{w}$ defined in (1).