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An efficient numerical method for singular perturbation problems

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Abstract

In this paper, we propose a method for the numerical solution of singularly perturbed two-point boundary-value problems (BVPs). First, we develop two schemes to integrate initial–value problem (IVP) for system of two first-order differential equations, and then by using these schemes we solve the BVP. Precisely, we convert the second-order BVP into a system of first-order differential equations, and then apply the numerical schemes to obtain the solution. In order to get an initial condition for the system, we use the asymptotic approximate solution. Error estimates are derived and numerical examples are provided to illustrate the present method.

Keywords: Singularly perturbed differential equations; Asymptotic approximation; Exponentially fitted difference scheme; Adaptive single-step exponential method

1. Introduction

Singular perturbation problems (SPPs) are of common occurrence in many branches of applied mathematics such as fluid dynamics, elasticity, chemical reactor theory, etc. It is a well-known fact that the solution of SPPs exhibits a multi-scale character; that is, there are thin layer(s) where the solution varies rapidly, while away from the layer(s) the solution behaves regularly and varies slowly. So the numerical treatment of singularly perturbed differential equations (DEs) gives major computational difficulties, and in recent years, a large number of special purpose methods have been developed to provide accurate

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numerical solutions. For details, one may refer to the books of Farrell et al. [2], Roos et al. [10], Miller et al. [4] and Morton [5] and the references therein.

Consider the following singularly perturbed two-point boundary-value problem (BVP):

$$Lu(x) \equiv \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in D = (0, 1), \tag{1.1}$$

$$B_0 u(0) \equiv b(0)u(0) - a(0)u'(0) = -f(0), \quad B_1 u(1) \equiv u(1) = B, \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter, a, b and f are smooth functions such that $a(x) \geqslant a > 0$, $b(x) \geqslant b \geqslant 0$, $x \in \overline{D} = [0, 1]$. Under these assumptions the BVP ((1.1)–(1.2)) has a unique solution $u(x)$ exhibiting a 'less severe boundary layer at $x = 0$ [1,10]. The 'less severe' means the solution $u(x)$ of the BVP ((1.1)–(1.2)) and its first derivative are bounded uniformly, for all ε on the interval [0, 1]. It may be noted that the reduced problem satisfies the boundary condition at $x = 0$ exactly.

Roberts [9] proposed a method known as boundary-value technique and introduced the idea of inner and outer solution regions for the domain [0, 1]. We suggested a similar technique in [6] for singularly perturbed turning point problems. In [8,12], we proposed domain decomposition methods for the BVP ((1.1)–(1.2)) which suit well for parallel computers.

In this paper, we develop two schemes to integrate singularly perturbed system of initial-value problems (IVPs); the first method is a combination of the classical finite difference scheme and the exponentially fitted difference (EFD) scheme of Doolan et al. [1] which is of order $O(h)$. The second method is derived by following the underlying idea of the schemes given in Vigo-Aguiar and Ferrándiz [11]. This new scheme is of order $O(h^2)$ and the accuracy of the method mainly depends on the initial-values. This scheme integrates exactly the differential equation with constant coefficients without local truncation error. These two schemes have been applied to the given BVP ((1.1)–(1.2)) after converting it into a system of IVPs in the whole domain D like the classical shooting method.

For the approximation of the initial-value we use the asymptotic approximate solution, which is an $O(\varepsilon)$ or higher-order approximation to the exact solution. By this one can reduce the number of iterations in the shooting technique by adding or subtracting a small increment to this asymptotic approximate solution to obtain a significant accuracy to the numerical solution at the other end of the interval. In general, IVPs are easier to handle than BVPs, in the sense that one can use more number of mesh points to integrate the DE. Moreover, the schemes presented here are having exponential weights to control the fast growth or decay in the exact solution because of the presence of the small parameter ε, and it avoids the use of very small step-size relative to ε.

The first method which is a combination of the EFD and classical schemes is presented in Section 2. Section 3 deals with the adaptive single-step exponential method. Error estimates are derived in Section 4. Numerical examples are given in Section 5.

Throughout this paper C will denote an arbitrary constant independent of mesh points x_i, mesh size h and the singular perturbation parameter ε. We define $\|\cdot\|$ of $\mathbf{w} = (w_1, w_2)^{\mathrm{T}} \in \mathbb{R}^2$ as $\|\mathbf{w}\| = \max\{|w_1|, |w_2|\}$.

2. Classical and exponentially fitted scheme

In this section, we present a numerical scheme for IVP for system of two first-order DEs, which is a combination of the classical upwind scheme and the EFD scheme of Doolan et al. [1] for single first-order IVP.