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A streamline tracking algorithm for semi-Lagrangian advection schemes based on the analytic integration of the velocity field

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Abstract

A new scheme for the construction on an unstructured grid of the streamlines of the three-dimensional shallow water equations is presented. The qualitative advantages of the scheme, notably closed streamlines and realistic treatment of closed boundaries, are derived and the spatial accuracy is demonstrated.

Semi-Lagrangian advection schemes offer the computational cost advantage of being explicit but also unconditionally stable with respect to time step. However, semi-Lagrangian methods based on the numerical integration of the discretised velocity field frequently have difficulty in meeting physically significant criteria such as the closure of streamlines and the inviolability of closed boundaries. Here a streamline tracking scheme based on the analytic integration of the discretised velocity field is presented.

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Keywords: Computational fluid dynamics; Semi-Lagrangian; Advection; Streamlines; Finite volume; Unstructured mesh

1. Introduction

The streamline tracking scheme presented here has been developed as a part of a scheme for the three-dimensional shallow water equations [3]. The mesh used is composed of triangular prisms and

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the variables are discretised in a generalised Arakawa C grid scheme with the cell face normal velocity component at the centre of each face. The continuity equation is discretised using a finite volume scheme so the integral of the divergence of the velocity over a cell is guaranteed to be zero. A feature of this discretisation is that if a single layer of cells is used, then the scheme reduces to a solver for the two-dimensional shallow water equations. The tracking of streamlines in two-dimensional flow is therefore also of interest.

The concept behind a semi-Lagrangian scheme is that the material derivative of the velocity may be discretised by tracing a streamline back through one time step to calculate where the fluid that will arrive at a given point at the end of a time step has come from. The focus of this paper is on a method for constructing the streamlines rather than on the interpolation of the advected velocity. The importance of the quality of streamline tracking has been recognised elsewhere as significant in the results of semi-Lagrangian advection algorithms [4].

2. Streamlines

The streamlines of a velocity field are obtained by solving the ordinary differential equation:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}). \tag{1}$$

It is an obvious requirement that the continuous velocity field used to calculate the streamlines be consistent with the discretised velocity field produced by the flow model. Since the flow model is intended to model physical behaviour it is desirable that the streamlines produced should exhibit physically realistic behaviour. In particular, we shall require that the streamlines produced not cross each other or a closed boundary (the no crossing condition).

3. Integration techniques

The conventional way to integrate the velocity field would be using a numerical ODE solver (see, for example [4]). However, due to the errors inherent in numerical integration techniques, it is difficult or impossible to guarantee the no crossing condition. At a minimum, very small time steps are required when calculating streamlines near closed boundaries. Instead, we generalise the approach in [2] to our unstructured grid. The basis for this approach is the construction of a continuous velocity field which is then integrated analytically to produce an analytic expression for the streamline starting at a given point. Since the integration is analytic, no errors (up to machine precision) are introduced by the integration process.

The analytic integration approach imposes another constraint on the continuous velocity field: it must be analytically integrable at a reasonable computational cost. The obvious candidate is a cell-wise linear field. Assume $\mathbf{x}$ lies in cell i then:

$$\mathbf{u}(\mathbf{x}) = A_i\mathbf{x} + b_i, \tag{2}$$

where A_i is a constant matrix and $\mathbf{b}_i$ is a constant vector. The integration problem reduces in this case to the solution of a three-dimensional system of linear ordinary differential equations. To determine the unknown matrix and vector, we first impose the constraints given by the discretised flow field. For each