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Journal of Computational and Applied Mathematics 195 (2006) 134–154
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

# A new asymptotic series for the Gamma function ☆

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Received 15 August 2004

**Abstract**

The famous Stirling's formula says that $\Gamma(s+1)=\sqrt{2\pi s}(s/e)^s e^{\gamma(s)}=\sqrt{2\pi}(s/e)^s e^{\theta(s)/12s}$. In this paper, we obtain a novel convergent asymptotic series of $\gamma(s)$ and proved that $\theta(s)$ is increasing for $s>0$.


*MSC:* 33B15; 05A16

*Keywords:* Gamma function; Asymptotic series; Stirling formula

## 1. Introduction

The Gamma function, one of the most famous functions in both mathematics and applied sciences,

$$s! = \Gamma(s+1) = \int_0^\infty x^s e^{-x}\,\mathrm{d}x, \qquad s>0 \tag{1}$$

can be analytically expanded to the whole complex plane excluding non-positive integers. It is well-known that the Gamma function has the following famous Stirling asymptotic series

☆ This paper is partly funded by ARO (DAAD19-03-1-0375) and Multimedia and Intelligent Software Technology Laboratory of Beijing University of Technology (KP0706200378).
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doi:10.1016/j.cam.2005.03.081

(see [1, p. 167, 6, pp. 111–112])

$$\ln(s-1)! = \frac{1}{2}\ln 2\pi + \left(s-\frac{1}{2}\right)\ln s - s + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n}{2n(2n-1)s^{2n-1}}, \tag{2}$$

where $B_m$ are Bernoulli numbers. However, series (2) is not convergent (see [1, p. 167]). In addition, Lanczos obtained an efficient method for numerical calculation of the Gamma function (see [2]). In this paper we obtained the following novel convergent asymptotic series.

**Theorem 1.** *Let $v \geqslant 0$ be a real number and $s$ be a complex number such that $Re(s) \geqslant 1$, where $Re(s)$ denotes the real part of $s$. Then, we obtained*

$$\ln s! = \tfrac{1}{2}\ln 2\pi + (s+\tfrac{1}{2})\ln s - s + \gamma(s), \tag{3}$$

*where*

$$\gamma(s) = \sum_{k=1}^{\infty}\frac{a_k(v)}{(s+v)(s+v+1)\cdots(s+v+k-1)},$$

$$a_k(v) = \frac{1}{2k}\int_0^1 (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t.$$

Set $v=1$ in (3), we obtain the Binet's formula (see [7, p. 253])

$$\ln(s-1)! = \frac{1}{2}\ln 2\pi + \left(s-\frac{1}{2}\right)\ln s - s + \sum_{k=1}^{\infty}\frac{a_k(1)}{(s+1)(s+2)\cdots(s+k)}. \tag{4}$$

Furthermore, we generalize equality (3) to all complex numbers $s$ excluding negative integers.

**Corollary 1.** *For all complex numbers $s$ but not negative integer, let $m \geqslant -[Re(s)]$ be a non-negative integer, where $[x]$ is defined to be the greatest integer not exceeding $x$. Then, we have*

$$\ln s! = \frac{1}{2}\ln 2\pi + \left(s+m+\frac{1}{2}\right)\ln(s+m+1) - \sum_{p=1}^{m}\ln(s+p)$$
$$- s - m - 1 + \gamma(s+m+1). \tag{5}$$

For the polynomials $a_k(v)'s$ introduced in Theorem 1, there holds the following results.

**Theorem 2.** *For any $v \geqslant 0$, $a_1(v) = \frac{1}{12}$, and $a_2(v) = v/12$. If $v \geqslant v_0 \approx 0.146094$, the positive root of $240v^3 + 600v^2 + 270v - 53 = 0$, then $a_k(v) > 0$ for $k \geqslant 3$. If $v = 0$, $a_k(v) < 0$ for $k \geqslant 3$. If $0 < v < v_0$, then there exists an integer $K$, such that $a_k(v) > 0$ for $k \geqslant K$.*

In this paper, for any complex number $s$ with $Re(s) \geqslant 1$, we obtain the following generalized Stirling formula in a new approach (see [7, p. 253])

$$s! = \sqrt{2\pi s}\left(\frac{s}{e}\right)^s e^{\gamma(s)} = \sqrt{2\pi}\left(\frac{s}{e}\right)^s e^{\theta(s)/12s}, \tag{6}$$

where $\theta(s)$ is an analytic function satisfying $0 < \theta(s) < 1$ for real numbers $s \geqslant 1$.