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Journal of Computational and Applied Mathematics 195 (2006) 292–299
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

A dual functional to the univariate B-spline☆

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Received 15 August 2004; received in revised form 16 March 2005

Abstract

In this paper, dual functionals with local supports to the univariate B-splines are constructed by symmetrization, which are a linear combination of function values.

Keywords: B-spline; Blossom; Dual basis; Polarization identity; Symmetrization

1. Introduction

Let $t := (t_i)$ be nondecreasing with $t_i < t_{i+n+1}$ for all i. The B-spline N_i of order $n+1$ and knots $t_i, t_{i+1}, \ldots, t_{i+n+1}$ is defined by

$$N_i(t) = (t_{i+n+1} - t_i)[t_i, \ldots, t_{i+n+1}](\cdot - t)^n_+.$$

Since $(x-t)^n = (x-t)^n_+ + (-1)^n (t-x)^n_+$, $N_i(t) = (-1)^{n+1}(t_{i+n+1} - t_i)[t_i, \ldots, t_{i+n+1}](t - \cdot)^n_+$. For a polynomial $p(t) = a_0 + a_1 t + \cdots + a_n t^n = \sum_{k=0}^n a_k t^k$, its blossom is defined as follows:

$$B(p)(t_1, \ldots, t_n) = \sum_{k=0}^{n} a_k \frac{t_*^k}{c_n^k},$$

☆ This research was partially supported by Chinese NSF Tianyuan Foundation A0324655 and Chinese NSF 60275029.
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doi:10.1016/j.cam.2005.03.088

where c_n^m is a combinatory number and will be used hereafter and $t_*^0, t_*^1, \ldots, t_*^n$ are the elementary symmetric polynomials of $t_0, t_1, \ldots, t_n$

$$t_*^0 = 1,\ \ t_*^1 = t_1 + \cdots + t_n,$$

$$t_*^2 = t_1 t_2 + t_1 t_3 + \cdots + t_{n-1} t_n, \ldots, t_*^n = t_1 t_2 \cdots t_n.$$

In [3], the following polarization identity is given:

$$B(p)(t_1, \ldots, t_n) = \frac{1}{n!} \sum_{S \subseteq \{1,2,\ldots,n\}, |S|=k} (-1)^{n-k} k^n p\left(\frac{1}{k}\sum_{i \in S} t_i\right).$$

Marsden's identity plays an important role in the construction of approximation operators of splines. It is as follows: for any $k, n \in N$, and for any knot sequence $(t_i)_{i=-\infty}^{+\infty}$, $(x-t)^n = \sum_i \omega_i(x) N_i(t)$, $\omega_i(x) = \prod_{k=i+1}^{i+n} (x - t_k)$, or

$$p(t) = \sum_{k=0}^{n} a_k t^k = \sum_i B(p)(t_{i+1}, \ldots, t_{i+n}) N_i(t).$$

From Marsden's identity, the following well-known results with respect to de Boor–Fix functionals can easily be obtained [2].

Theorem 1.1. *Any polynomial of degree at most n has the B-spline expansion*

$$p(t) = \sum_i \lambda_i(p) N_i(t),$$

where

$$\lambda_i(p) = \lambda_i(p, x) = \sum_{m=1}^{n+1} (-1)^{n-m+1} \psi_i^{(m-1)}(x) p^{(n-m+1)}(x) \quad \forall x \in R,$$

$$\psi_i(x) = \frac{\omega_i(x)}{n!}.$$

Any spline $s(t) = \sum_i c_i N_i(t)$ *has the B-spline expansion*

$$s(t) = \sum_i \lambda_i(s, \xi_i) N_i(t) \quad \forall \xi_i \in (t_i, t_{i+n+1}).$$

The latter de Boor–Fix functionals are dual functionals with local support to B-splines, i.e. supp $\lambda_i \subseteq [t_i, t_{i+n+1}]$, $\lambda_i(N_j) = \delta_{ij}$.

A direct construction of the functionals (λ_i) dual to the B-spline basis (N_i) may proceed as follows: given $\Delta = (t_i)$, $t_i \in [a, b]$, let h_i be a function such that $h_i \in C^{n+1}[a; b]$ and

$$h_i(t_j) = 0, \quad j \leqslant i + n; \quad h_i(t_j) = \psi_i(t_j), \quad j \geqslant i + 1,$$