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Hook-lengths and pairs of compositions

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Abstract

The monomial basis for polynomials in N variables is labeled by compositions. To each composition there is associated a hook-length product, which is a product of linear functions of a parameter. The zeroes of this product are related to "critical pairs" of compositions; a concept defined in this paper. This property can be described in an elementary geometric way; for example: consider the two compositions $(2, 7, 8, 2, 0, 0)$ and $(5, 1, 2, 5, 3, 3)$, then the respective ranks, permutations of the index set $\{1,2,\ldots,6\}$ sorting the compositions, are $(3, 2, 1, 4, 5, 6)$ and $(1, 6, 5, 2, 3, 4)$, and the two vectors of differences (between the compositions and the ranks, respectively) are $(-3, 6, 6, -3, -3, -3)$ and $(2, -4, -4, 2, 2, 2)$, which are parallel, with ratio $-\frac{3}{2}$. For a given composition and zero of its hook-length product there is an algorithm for constructing another composition with the parallelism property and which is comparable to it in a certain partial order on compositions, derived from the dominance order. This paper presents the motivation from the theory of nonsymmetric Jack polynomials and the description of the algorithm, as well as the proof of its validity.

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1. Introduction

A composition is an element of $\mathbb{N}_0^N$ (where $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$); a typical composition is $\alpha = (\alpha_1, \ldots, \alpha_N)$ and the components α_i are called the parts of α. Compositions have the obvious application of labeling the monomial basis of polynomials in the variables $x_1, \ldots, x_N$ and they also serve as labels for the nonsymmetric Jack polynomials (a set of homogeneous polynomials which are simultaneous eigenfunctions of a certain parametrized and commuting set $\{\mathscr{U}_i : 1 \leqslant i \leqslant N\}$ of difference–differential operators). In this context the ranks of the parts of a composition become significant. The ranks are based on sorting on magnitude and index so that the largest part has rank 1; if a value is repeated then the one with lower index has lower rank. This is made precise in the following (the cardinality of a set E is denoted by $\#E$):

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Definition 1. For $\alpha \in \mathbb{N}_0^N$ and $1 \leqslant i \leqslant N$ let $r(\alpha, i) := \#\{j : \alpha_j > \alpha_i\} + \#\{j : 1 \leqslant j \leqslant i, \alpha_j = \alpha_i\}$ be the rank function.

A consequence of the definition is that $r(\alpha, i) < r(\alpha, j)$ is equivalent to $\alpha_i > \alpha_j$, or $\alpha_i = \alpha_j$ and $i < j$. For any α the function $i \mapsto r(\alpha, i)$ is one-to-one on $\{1, 2, \ldots, N\}$. A partition is a composition satisfying $\alpha_i \geqslant \alpha_{i+1}$ for all i, equivalently, $r(\alpha, i) = i$ for all i. For a fixed $\alpha \in \mathbb{N}_0^N$ the values $\{r(\alpha, i) : 1 \leqslant i \leqslant N\}$ are independent of trailing zeros, that is, if $\alpha' \in \mathbb{N}_0^M$, $\alpha'_i = \alpha_i$ for $1 \leqslant i \leqslant N$ and $\alpha'_i = 0$ for $N < i \leqslant M$ then $r(\alpha, i) = r(\alpha', i)$ for $1 \leqslant i \leqslant N$, and $r(\alpha', i) = i$ for $N < i \leqslant M$. A formal parameter κ appears in the construction of nonsymmetric Jack polynomials; their coefficients are in $\mathbb{Q}(\kappa)$, a transcendental extension of $\mathbb{Q}$. The relevant information in a composition label is encoded as the function $i \mapsto \alpha_i - \kappa r(\alpha, i)$. We will be concerned with situations where a pair (α, β) of compositions has the property that $\alpha_i - \kappa r(\alpha, i) = \beta_i - \kappa r(\beta, i)$ for all i, when κ is specialized to some negative rational number. This is equivalent to the condition that $(r(\beta, i) - r(\alpha, i))\kappa + \alpha_i - \beta_i$ is a rational multiple of $m\kappa + n$ for some fixed $m, n > 0$ (or that the vectors $(\alpha_i - \beta_i)_{i=1}^N$ and $(r(\alpha, i) - r(\beta, i))_{i=1}^N$ are parallel). For our application an additional condition is imposed on the pair (α, β) which is stated in terms of a partial order on compositions. Let S_N denote the symmetric group on N objects, considered as the permutation group of $\{1, 2, \ldots, N\}$. The action of S_N on compositions is defined by $(w\alpha)_i = \alpha_{w^{-1}(i)}$, $1 \leqslant i \leqslant N$.

Definition 2. For a composition $\alpha \in \mathbb{N}_0^N$ let $|\alpha| := \sum_{i=1}^N \alpha_i$ and let $\ell(\alpha) := \max\{j : \alpha_j > 0\}$ be the length of α.

Definition 3. For $\alpha \in \mathbb{N}_0^N$ let α^+ denote the unique partition such that $\alpha^+ = w\alpha$ for some $w \in S_N$. For $\alpha, \beta \in \mathbb{N}_0^N$ the partial order $\alpha \succ \beta$ (α dominates β) means that $\alpha \neq \beta$ and $\sum_{i=1}^j \alpha_i \geqslant \sum_{i=1}^j \beta_i$ for $1 \leqslant j \leqslant N$; and $\alpha \rhd \beta$ means that $|\alpha| = |\beta|$ and either $\alpha^+ \succ \beta^+$ or $\alpha^+ = \beta^+$ and $\alpha \succ \beta$.

For a given $\alpha \in \mathbb{N}_0^N$ let w be the inverse function of $i \mapsto r(\alpha, i)$ then $r(\alpha, w(j)) = j$ for $1 \leqslant j \leqslant N$ and $\alpha = w\alpha^+$. This permutation appears again in part (iii) of Proposition 3.

Definition 4. A pair (α, β) of compositions is a $(-n/m)$-critical pair (where $m, n \geqslant 1$) if $\alpha \rhd \beta$ and $m\kappa + n$ divides $(r(\beta, i) - r(\alpha, i))\kappa + \alpha_i - \beta_i$ (in $\mathbb{Q}[\kappa]$) for each i.

The divisibility property is equivalent to $(r(\beta, i) - r(\alpha, i))n = m(\alpha_i - \beta_i)$ for all i. By elementary arguments we show why only negative numbers appear in the critical pairs, and we also find a bound on $\ell(\beta)$. A simple example shows that $m = 0$ is possible: let $\alpha = (3, 0)$ and $\beta = (2, 1)$, then both α and β have ranks $(1, 2)$.

Proposition 1. *Suppose* $\alpha, \beta \in \mathbb{N}_0^N$, $\alpha \rhd \beta$ *and there are integers* m, n *such that* $((r(\beta, i) - r(\alpha, i))\kappa + \alpha_i - \beta_i)/(m\kappa + n) \in \mathbb{Q}$ *for* $1 \leqslant i \leqslant N$, *then* $mn \geqslant 0$ *and* $n \neq 0$.

Proof. The case $n = 0$ is impossible since that would imply $\alpha_i - \beta_i = 0$ for all i, that is, $\alpha = \beta$. So we assume $n \geqslant 1$ and then show $m \geqslant 0$. Let w be the inverse function of $i \mapsto r(\alpha, i)$ (so that $r(\alpha, w(i)) = i$). By definition either $\alpha^+ \succ \beta^+$ or $\alpha^+ = \beta^+$ and $\alpha \succ \beta$. Suppose that $\alpha^+ \succ \beta^+$ and let $k \geqslant 1$ have the property that $\beta_{w(j)} = \alpha_{w(j)}$ and $r(\beta, w(j)) = j$ for $1 \leqslant j < k$ and at least one of $\beta_{w(k)} \neq \alpha_{w(k)}$ and $r(\beta, w(k)) > k = r(\alpha, w(k))$ holds. Define l by $r(\beta, l) = k$, then by the definition of the dominance order $\succ$ we have that $\alpha_{w(k)} \geqslant \beta_l$. Also $\beta_l \geqslant \beta_{w(k)}$ because $r(\beta, w(k)) \geqslant k$. The case $\alpha_{w(k)} = \beta_{w(k)}$ and $r(\beta, k) > k$ (thus $n = 0$) is impossible hence $\alpha_{w(k)} > \beta_{w(k)}$. If $r(\beta, k) = k$ then $m = 0$ or else $r(\beta, k) > k$ and $m > 0$.

Now suppose $\alpha^+ = \beta^+$ and $\alpha \succ \beta$, and let $k \geqslant 1$ have the property that $\beta_j = \alpha_j$ for $1 \leqslant j < k$ and $\alpha_k > \beta_k$ (the existence of k follows from the definition of $\alpha \succ \beta$). Since β is a permutation of α we have that $r(\alpha, j) = r(\beta, j)$ for $1 \leqslant j < k$ and $r(\alpha, k) < r(\beta, k)$. This implies $m > 0$. $\square$

Proposition 2. *Suppose that* (α, β) *is a* $(-n/m)$*-critical pair, for some* $m, n \geqslant 1$, *then* $\ell(\beta) \leqslant \ell(\alpha) + |\alpha|$.

Proof. First we show that if $i > \ell(\alpha)$ and $\beta_i = 0$ then $\beta_j = 0$ for all $j > i$. By hypothesis $(m\kappa + n)$ divides $(r(\beta, i) - r(\alpha, i))\kappa + (\alpha_i - \beta_i) = (r(\beta, i) - i)\kappa$, hence $r(\beta, i) = i$. This implies that $0 \leqslant \beta_j \leqslant \beta_i = 0$ for all $j > i$. Thus if $\ell(\beta) > \ell(\alpha)$ then $\beta_i \geqslant 1$ for $\ell(\alpha) < i \leqslant \ell(\beta)$. Since $|\beta| = |\alpha|$ this shows that $\ell(\beta) - \ell(\alpha) \leqslant |\alpha|$. $\square$