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Separation theorems for the zeros of certain hypergeometric polynomials

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Abstract

We study interlacing properties of the zeros of two contiguous ${}_2F_1$ hypergeometric polynomials. We use the connection between hypergeometric ${}_2F_1$ and Jacobi polynomials, as well as a monotonicity property of zeros of orthogonal polynomials due to Markoff, to prove that the zeros of contiguous hypergeometric polynomials separate each other. We also discuss interlacing results for the zeros of ${}_2F_1$ and those of the polynomial obtained by shifting one of the parameters of ${}_2F_1$ by $\pm t$ where $0 < t < 1$.

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1. Introduction

The general hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; x\right) = 1 + \sum_{k=1}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k x^k}{(b_1)_k (b_2)_k \ldots (b_q)_k k!}, \quad |x| < 1, $$

where

$$ (\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1), \quad k \geqslant 1, \quad k \in \mathbb{N} $$

is Pochhammer's symbol.

If one of the numerator parameters is equal to a negative integer, say $a_1 = -n$, $n \in \mathbb{N}$, then the series terminates and is a polynomial of degree n in x. A natural, and often very important, question that arises in the study of polynomials is

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the nature and location of their zeros. The connection between ${}_2F_1$ hypergeometric polynomials and orthogonal polynomials, particularly the Jacobi polynomials, leads to fairly complete information about the zeros of ${}_2F_1$ polynomials. However, there is no general link between ${}_3F_2$ polynomials and classical orthogonal polynomials and the investigation of the location of the zeros of ${}_3F_2$ polynomials is far more challenging.

Some classes of ${}_3F_2$ polynomials can be expressed as products of ${}_2F_1$ polynomials (cf. [5, p. 498]) and the location of their zeros in various intervals on the real line can be deduced (cf. [3,4]). When generating numerical data for the zeros of these classes of ${}_3F_2$ polynomials, it appeared that in some cases the real, simple zeros of the ${}_2F_1$ polynomials which are factors of the ${}_3F_2$ polynomials are interlacing. This observation posed the question of when and how interlacing of the zeros of ${}_2F_1$ polynomials takes place.

In this paper, we examine the question of the interlacing properties of the zeros of so-called contiguous ${}_2F_1$ hypergeometric polynomials. The six functions

$$ {}_2F_1\left(\begin{matrix} a\pm 1, b \\ c \end{matrix}; x\right), \quad {}_2F_1\left(\begin{matrix} a, b\pm 1 \\ c \end{matrix}; x\right) \quad \text{and} \quad {}_2F_1\left(\begin{matrix} a, b \\ c\pm 1 \end{matrix}; x\right) $$

are contiguous to

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) $$

(cf. [6]) and there are identities that link

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) $$

with any pair of its contiguous functions via a linear relation in x (cf. [6, p. 72]). We shall use the known interlacing properties of Jacobi polynomials of successive degrees and also the monotonicity result of Markoff (cf. [7, p. 116, Theorem 6.12.2]) to prove our results.

The Jacobi polynomials $\mathscr{P}_n^{(\alpha,\beta)}(x)$ and ${}_2F_1$ polynomials are linked by (cf. [1, p. 295])

$$ \mathscr{P}_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, 1+\alpha+\beta+n \\ 1+\alpha \end{matrix}; \frac{1-x}{2}\right). $$

For $\alpha, \beta > -1$, the sequence $\{\mathscr{P}_n^{(\alpha,\beta)}(x)\}_{n=1}^{\infty}$ of Jacobi polynomials is orthogonal on $(-1, 1)$ with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ (cf. [1, p. 299, Theorem 6.4.3]). Correspondingly,

$$ {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right) $$

is orthogonal to all polynomials of lower degree with respect to the (varying with n) weight function $x^{c-1}(1-x)^{b-c-n}$ on $(0, 1)$ for $c > 0$ and $b > c + n - 1$; on $(1, \infty)$ for $b < 1 - n$ and $c < b + 1 - n$ with respect to weight function $x^{c-1}(x-1)^{b-c-n}$; and with respect to $(-x)^{c-1}(1-x)^{b-c-n}$ on $(-\infty, 0)$ for $b < 1 - n$ and $c > 0$ (cf. [2]).

We shall assume throughout our discussion that $b, c \in \mathbb{R}$ and denote the functions contiguous to

$$ F_n(x) = {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right) $$

by $F_n(b+; x)$ etc. where

$$ F_n(b+; x) = {}_2F_1\left(\begin{matrix} -n, b+1 \\ c \end{matrix}; x\right). $$

2. Separation theorems for the zeros of contiguous hypergeometric polynomials

Our first result is just a direct translation of the known interlacing properties of the zeros of the Jacobi polynomials $\mathscr{P}_n^{(\alpha,\beta)}(x)$ and $\mathscr{P}_{n+1}^{(\alpha,\beta)}(x)$ for $\alpha, \beta > -1$ (cf. [1, p. 253, Theorem 5.4.2]) to the corresponding ${}_2F_1$ hypergeometric polynomials.