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JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Journal of Computational and Applied Mathematics 197 (2006) 62 – 77

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Semi-cardinal interpolation and difference equations: From cubic B-splines to a three-direction box-spline construction

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Received 8 July 2005
Available online 8 July 2005

Abstract

This paper considers the problem of interpolation on a semi-plane grid from a space of box-splines on the three-direction mesh. Building on a new treatment of univariate semi-cardinal interpolation for natural cubic splines, the solution is obtained as a Lagrange series with suitable localization and polynomial reproduction properties. It is proved that the extension of the natural boundary conditions to box-spline semi-cardinal interpolation attains half of the approximation order of the cardinal case.

MSC: primary 41A15; secondary 41A05; 41A25; 41A63; 39A70; 47B35

Keywords: Multivariable interpolation; Box-splines; Boundary conditions; Difference equations; Wiener–Hopf

1. Introduction

A treatment of cardinal interpolation (i.e. interpolation at the set $\mathbb{Z}$ of integers) with univariate polynomial splines was given by Schoenberg in [26], using the concept of B-spline functions. In [25,27], Schoenberg also considered the related problem of semi-cardinal interpolation (i.e. interpolation at the set $\mathbb{Z}_+$ of non-negative integers) from a space of univariate splines satisfying certain end-point conditions. The extension of cardinal interpolation to three-directional bivariate box-splines interpolating data on the grid $\mathbb{Z}^2$ was obtained by de Boor et al. [10].

The present paper introduces the problem of interpolation on the semi-plane grid $\mathbb{Z} \times \mathbb{Z}_+$ from a space of bivariate piecewise polynomial functions generated by the three-direction box-spline M whose direction matrix has every multiplicity 2. This box-spline represents a genuine bivariate analog of the univariate cubic B-spline, and its utility for approximation and computer aided design has been established early in several studies by Frederickson [12–14], Sabin [20,21] Sablonnière [22–24], Chui and Wang [6].

The main idea of our bivariate extension of semi-cardinal interpolation to box-splines is to formulate certain automatic boundary conditions in terms of finite difference equations for box-spline coefficients. As demonstrated by Chui et al. [5], the presence of boundary conditions complicates the study of spline spaces even for simpler generating functions than M. Bringing in Fourier methods from the theory of bi-dimensional Wiener-Hopf difference equations, the model

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2005.10.026

of interpolation on a semiplane grid enables a full analysis of localization and polynomial reproduction properties of the proposed box-spline scheme.

Our approach is first illustrated for the univariate case in Section 2, in which the space of one-dimensional semi-cardinal cubic splines is regarded as a subspace of cardinal splines whose B-spline coefficients satisfy a system of ‘natural’ difference equations. The resulting construction is simpler than those obtained by Schoenberg in [25,27]. In Section 3, we propose a suitable extension of the natural boundary conditions to bivariate piecewise polynomials generated by M. The corresponding semi-cardinal interpolation problem with box-splines is then solved by constructing the set of fundamental functions and the associated Lagrange scheme. Our analysis is based on the explicit solution of bi-dimensional difference equations of Wiener–Hopf type. In Section 4 we prove that the ‘natural’ semi-cardinal box-spline scheme attains half of the approximation order of the corresponding cardinal scheme. The generalization of these results to other box-splines requires significantly different methods of proof in order to avoid explicit computations. This remains a problem for future research.

Note that a different multivariate extension of semi-cardinal interpolation was obtained in [2,3] for certain polyharmonic spline methods, using a Fourier transform treatment [1] of the univariate case. However, a complete analysis establishing the approximation order of the polyharmonic semi-cardinal schemes has yet to be achieved.

Notation: For a given integer n, the set of integers smaller than or equal to n will be denoted by $\mathbb{Z}_{\leqslant n}$. Also, $\mathbb{Z}_{\geqslant n} := \mathbb{Z}\setminus\mathbb{Z}_{\leqslant n-1}$, $\mathbb{Z}_+ := \mathbb{Z}_{\geqslant 0}$, and $\mathbb{R}_+ := [0, \infty)$.

2. Semi-cardinal interpolation with cubic B-splines

A *cardinal cubic spline* is a function $s : \mathbb{R} \to \mathbb{C}$, such that

(i) $s \in C^2(\mathbb{R})$, and
(ii) s is a cubic polynomial on $[k, k+1]$, for any $k \in \mathbb{Z}$.

The space of such functions will be denoted by $\mathscr{S}_3$. In [27], Schoenberg studies interpolation to data given on $\mathbb{Z}_+$—referred to as ‘semi-cardinal interpolation’—from the linear space $\mathscr{S}_3^+$ of functions $s : \mathbb{R} \to \mathbb{C}$ satisfying (i), as well as

(iii) s is a cubic polynomial on $[k, k+1]$, for any $k \in \mathbb{Z}_+$, and
(iv) $s''(x) = 0$, for $x \in (-\infty, 0]$ (i.e. s is a linear polynomial on $(-\infty, 0]$).

Since (iv) is known as a ‘natural’ end condition, an arbitrary element of $\mathscr{S}_3^+$ will be called a *natural semi-cardinal cubic spline*.

Two methods are used in [27] in order to construct semi-cardinal interpolation from $\mathscr{S}_3^+$ and, more generally, from similar odd-degree spline spaces. The first one [27, Chapter I] is based on a set of ‘fundamental functions’ $\{L_j : j \in \mathbb{Z}_+\} \subset \mathscr{S}_3^+$ satisfying

$$L_j(k) = \delta_{jk}, \quad j, k \in \mathbb{Z}_+, \tag{2.1}$$

such that the corresponding Lagrange scheme

$$s(x) = \sum_{j=0}^{\infty} y_j L_j(x), \quad x \in \mathbb{R}_+, \tag{2.2}$$

is absolutely and uniformly convergent on compact subsets of $\mathbb{R}_+$ for any data sequence $\{y_j\}_{j=0}^{\infty}$ of polynomial growth, and $s(j) = y_j$, $j \in \mathbb{Z}_+$. In turn, for each $j \in \mathbb{Z}_+$, the Lagrange function L_j is defined as a linear combination of the shifted fundamental function for cardinal interpolation and a set of so-called eigenspline functions [27, (4.2)]. It can be noted that this linear combination belongs to the cardinal space $\mathscr{S}_3$, but this fact is not explicitly used or mentioned by Schoenberg.

The second method [27, Chapter II] is employed for a different class of data sequences and builds on the fact that, if $s \in \mathscr{S}_3^+$, then the second derivative s'' is a cardinal spline of degree one, determined by its linear B-spline series.