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An adaptive version of a fourth-order iterative method for quadratic equations

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Abstract

A fourth-order iterative method for quadratic equations is presented. A semilocal convergence theorem is performed. A multiresolution transform corresponding to interpolatory technique is used for fast application of the method. In designing this algorithm we apply data compression to the linear and the bilinear forms that appear on the method. Finally, some numerical results are studied.

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1. Introduction

Determining the zeros of a nonlinear equation is a classical problem. These roots cannot in general be expressed in closed form. A powerful tool to study these equations is the use of iterative processes [17]. Starting from an initial guess x_0 successive approaches (until some predetermined convergence criterion is satisfied) x_i are computed, $i = 1, 2, \dots,$ with the help of certain iteration function $\Phi : X \to X$,

$$x_{n+1} := \Phi(x_n), \quad n = 0, 1, 2 \dots \tag{1}$$

In general, an iterative method is of pth order if the solution x^* of $F(x) = 0$ satisfies $x^* = \Phi(x^*)$, $\Phi'(x^*) = \cdots = \Phi^{p-1}(x^*) = 0$ and $\Phi^p(x^*) \neq 0$. For such a method, the error $\|x^* - x_{n+1}\|$ is proportional to $\|x^* - x_n\|^p$ as $n \to \infty$. It can be shown that the number of significant digits is multiplied by the order of convergence (approximately) by proceeding from x_n to x_{n+1}.

Newton's method and similar second-order methods are the most used [19]. Higher-order methods require more computational cost than other simpler methods, which makes them disadvantageous to be used in general, but, in some cases, it pays to be a little more elaborated.

In this paper, we are interested in the solution of quadratic equations

$$F(x) = 0, \tag{2}$$

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where $F : X \to Y$, X, Y Banach spaces and $F''(x)=B$ is a constant bilinear form. This is an example where third-order methods are a good alternative to Newton's type methods. Some particular cases of these type of equations, which appear in many applications, as control theory, are Riccati's equations [15].

In [12] the following family of third-order methods was introduced:

$$x_{n+1} = x_n - (I + \tfrac{1}{2}L_F(x_n)[I - \beta L_F(x_n)]^{-1})F'(x_n)^{-1}F(x_n), \tag{3}$$

where

$$L_F(x_n) = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n), \quad \beta \in [0, 1].$$

This family includes the classical Chebyshev ($\beta = 0$), Halley ($\beta = \frac{1}{2}$) and Super-Halley ($\beta = 1$) methods.

We refer to [12] and its references for a general convergence analysis of this type of third-order iterative methods. For the particular case of quadratic equations, we refer to [10], where the authors use the α-theory introduced in [18,20].

In general, third-order methods [1] can be written as

$$x_{n+1} = x_n - (I + \tfrac{1}{2}L_F(x_n) + \mathrm{O}(L_F(x_n)^2))F'(x_n)^{-1}F(x_n).$$

In particular, each iteration of a third-order method is between the iterations of two C-methods,

$$x_{n+1} = x_n - (I + \tfrac{1}{2}L_F(x_n) + CL_F(x_n)^2)F'(x_n)^{-1}F(x_n). \tag{4}$$

This class of scheme has been studied in [11,2].

In this paper, we analyze a C-method with fourth order of convergence for quadratic equations. The advantage of this method is that the matrix of the different associated linear systems in each iteration is the same.

On the other hand, multiresolution representations of data, such as wavelet decompositions, are useful tools for data compression. Given a finite sequence f^L, which represents sampling of weighted-averages of a function $f(x)$ at the finest resolution level L, multiresolution algorithms connect it with its multiscale representation $\{f^0, d^1, d^2, \ldots, d^L\}$, where the f^0 corresponds to the sampling at the coarsest resolution level and each sequence d^k represents the intermediate details which are necessary to recover f^k from f^{k-1}.

We consider the framework of Harten's multiresolution [13,14]. The greatest advantage of this general framework lies in its flexibility. For instance, boundary conditions receive a simplified treatment in this framework. Different types of setting can be considered depending on the linear operator that produces the data, we refer to [5,6] for more details. For simplicity, in this paper we consider the point value setting. In this setting, interpolation plays a key role. Usually, the interpolation is performed using polynomials.

One of the applications of multiresolution is matrix compression. If a matrix represents a smooth operator, its multiresolution representation can be transformed in a sparse matrix. This is called the standard form of a matrix [4,7]. We will need also a standard form for bilinear operators [3]. We use these standard forms to solve faster the linear system and the products appearing in the method.

The paper is organized as follows: The fourth-order iterative method is described and analyzed in Section 2, where we study a semilocal convergence theorem. We improve and simplify the hypothesis used in [11]. We recall in Section 3 the discrete pointvalue framework for multiresolution introduced by Harten [13,14] and the standard forms of a matrix and of a bilinear operator. We derive, using these multiresolution transforms, an adaptive version of the method. Finally, the algorithm is tested in Section 4 on several examples.

2. A fourth-order iterative method for quadratic equations

In general, for the C-methods (4) $\Phi'''(x^*) \neq 0$, so the cubic order of convergence cannot be reached. But if we take $C = C(x) = \frac{1}{2}(1 - L_{F'}(x)/3)$, we can improve the accuracy since in this case

$$\Phi'''(x^*) = 0.$$

For quadratic equations $L_{F'}(x) = 0$, and $C = C(x)$ will be the constant $C = \frac{1}{2}$.