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Asymptotic analysis of a perturbation problem

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Dedicated with admiration to Prof. Roderick S.C. Wong on the occasion of his 60th birthday

Abstract

An asymptotic expansion is constructed for the solution of the initial-value problem

$$u_{tt} - u_{xx} + u = \varepsilon(u_t - \tfrac{1}{3} u_t^3), \quad -\infty < x < \infty, \ t \geqslant 0,$$

$$u(x, 0) = \sin kx, \quad u_t(x, 0) = 0,$$

when t is restricted to the interval $[0, T/\varepsilon]$, where T is any given number. Our analysis is mathematically rigorous; that is, we show that the difference between the true solution $u(t, x; \varepsilon)$ and the Nth partial sum of the asymptotic series is bounded by ε^{N+1} multiplied by a constant depending on T but not on x and t.

Keywords: Nonlinear hyperbolic equations; Van der Pol-type perturbation; Multiple-scale method; Uniform asymptotic expansion

1. Introduction

Many physical problems are governed by nonlinear hyperbolic equations, where the nonlinear terms depend only on certain derivatives of the dependent variable and a small parameter ε. These weakly nonlinear problems are usually considered as perturbations of the corresponding linear hyperbolic equations. Since solutions of linear problems can be represented by superposition of uniform waves and solutions of nonlinear problems cannot usually be expressed explicitly, one often uses perturbation methods to represent the solutions of perturbed problems in the form of asymptotic series. If one starts with a regular

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perturbation expansion, i.e., a power series in ε, then a secular term appears in the second term of the expansion. Thus, this naive approach is useless for large values of t. To obtain an approximation for the solution, which holds uniformly for large t, one would have to use more sophisticated methods. The best known and most effective methods for this purpose are the Poincaré–Lindstedt's method and the multiple-scale method. Both of these methods are described in great detail in the books [2,6,8].

In the paper in [5], a van der Pol-type perturbation problem of the linear Klein–Gordon equation is studied. More precisely, the authors consider

$$u_{tt} - u_{xx} + u = \varepsilon(u_t - \tfrac{1}{3}u_t^3), \quad 0 \leqslant x \leqslant \pi, \ t \geqslant 0 \tag{1.1}$$

with boundary and initial conditions

$$u(0, t) = u(\pi, t) = 0 \tag{1.2}$$

and

$$u(x, 0) = f(x, \varepsilon), \quad u_t(x, 0) = g(x, \varepsilon), \tag{1.3}$$

respectively, where ε is a small positive parameter and f, g can, respectively, be represented by the Fourier sine series

$$f(x, 0) = \sum_{n=1}^{\infty} A_n(0) \sin nx \tag{1.4}$$

and

$$g(x, 0) = \sum_{n=1}^{\infty} \sqrt{n^2+1} B_n(0) \sin nx. \tag{1.5}$$

By using the method of two-timing (i.e., two scales), they derived the first term of a uniform asymptotic expansion of the solution $u(x, t; \varepsilon)$ for $t \leqslant \mathrm{O}(1/\varepsilon)$. The term obtained is given by

$$u_0(x, \sigma, \tau) = \sum_{n=1}^{\infty} \left[A_n(\tau) \cos\left(\sqrt{n^2+1}\sigma\right) + B_n(\tau) \sin\left(\sqrt{n^2+1}\sigma\right) \right] \sin nx, \tag{1.6}$$

where $\sigma = [1 + \mathrm{O}(\varepsilon^2)]t$, $\tau = \varepsilon t$,

$$\begin{pmatrix} A_n(\tau) \\ B_n(\tau) \end{pmatrix} = \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^{1/2} \exp\left\{ \frac{1}{2} \int_0^{\tau} y_n(s)\,\mathrm{d}s \right\} \begin{pmatrix} A_n(0) \\ B_n(0) \end{pmatrix}, \tag{1.7}$$

$$\mathrm{e}^{\tau} - 1 = \int_0^{z} \prod_{k=1}^{\infty} [1 - y_k(0)s]^{-4}\,\mathrm{d}s, \tag{1.8}$$

$$y_n(\tau) = y_n(0) \frac{\mathrm{d}z}{\mathrm{d}\tau} \frac{1}{1 - y_n(0)z} \tag{1.9}$$

and

$$y_n(0) = \frac{n^2+1}{16} [A_n^2(0) + B_n^2(0)]. \tag{1.10}$$