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Uniform asymptotic expansions of the Pollaczek polynomials

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Dedicated to Professor Roderick S.C. Wong on the occasion of his 60th birthday

Abstract

Two uniform asymptotic expansions are obtained for the Pollaczek polynomials $P_n(\cos\theta; a, b)$. One is for $\theta \in (0, \delta/\sqrt{n}]$, $0 < \delta < \sqrt{a+b}$, in terms of elementary functions and in descending powers of $\sqrt{n}$. The other is for $\theta \in [\delta/\sqrt{n}, \pi/2]$, in terms of a special function closely related to the modified parabolic cylinder functions, in descending powers of n. This interval contains a turning point and all possible zeros of $P_n(\cos\theta)$ in $\theta \in (0, \pi/2]$.

MSC: 41A60; 33C47

Keywords: Pollaczek polynomials; Uniform asymptotic expansion; Parabolic cylinder function; Airy function

1. Introduction

The Pollaczek polynomials $P_n(x; a, b)$ can be defined by the generating function

$$(1 - we^{i\theta})^{-(1/2)+ih(\theta)}(1 - we^{-i\theta})^{-(1/2)-ih(\theta)} = \sum_{n=0}^{\infty} P_n(x; a, b)w^n, \tag{1.1}$$

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where $x = \cos\theta$ for $\theta \in (0, \pi)$,

$$h(\theta) = \frac{a\cos\theta + b}{2\sin\theta}, \quad a > \pm b, \tag{1.2}$$

and each of the factors in the generating function reduces to 1 for $w = 0$.

The Pollaczek polynomials show in many aspects a singular behavior, see [10, pp. 393–396]. In his 1954 thesis [7], A. Novikoff investigated the asymptotic behavior of these polynomials and their zeros. He extracted the asymptotic behavior of $P_n(x; a, b)$ as $n \to \infty$, where $x = \cos(t/\sqrt{n})$, and $t > 0$ is fixed. He obtained the behavior in two intervals of t on both sides of $t = \sqrt{a+b}$. Furthermore, if the zeros of these polynomials are denoted by $\cos(\theta_{vn})$, where $0 < \theta_{1n} < \cdots < \theta_{nn} < \pi$, then he showed that for any fixed v,

$$\lim_{n\to\infty} \sqrt{n}\ \theta_{vn} = \sqrt{a+b}. \tag{1.3}$$

It was conjectured by R.A. Askey that the next term of the asymptotic expansion of θ_{vn} would involve a certain transcendental function. Partially inspired by this idea, Ismail [6] and Bo and Wong [4] both derived two term asymptotic expansions for θ_{vn}. Our approach in this paper bears some impact of [4], and is actually in accordance with Bleistein [3], Chester et al. [5] and Wong [12, Chapter VII].

One of the main results of Bo and Wong [4] is the following expansion

$$\theta_{vn} = \sqrt{\frac{a+b}{n}} + \frac{(a+b)^{1/6}(-a_v)}{2n^{5/6}} + \mathrm{O}\left(\frac{1}{n^{7/6}}\right), \tag{1.4}$$

where a_v is the vth negative zero of the Airy function. Since $-a_n = \mathrm{O}(n^{2/3})$ as $n \to \infty$ (cf. [10, p. 377]), we can see that (1.4) is unlikely to be applied to the largest zero. Indeed, (1.4) was derived based on a uniform asymptotic expansion for $\theta \in [\delta/\sqrt{n},\ M/\sqrt{n}]$. Despite the fact that this θ-interval includes the transition point $t = \sqrt{a+b}$, it includes only those extreme zeros. Hence it would be highly desirable to obtain a universal expansion in the whole interval $\theta \in (0, \pi)$. This is one of our main motivations of this investigation.

The present research is also motivated by the uniform treatment of Darboux's method. If we denote for short $P_n(x) = P_n(x; a, b)$, then

$$P_n(x) = \frac{1}{2\pi \mathrm{i}} \int_C (1 - w\mathrm{e}^{\mathrm{i}\theta})^{-(1/2)+\mathrm{i}h(\theta)} (1 - w\mathrm{e}^{-\mathrm{i}\theta})^{-(1/2)-\mathrm{i}h(\theta)} w^{-n-1}\, \mathrm{d}w, \tag{1.5}$$

where $x = \cos\theta$, C is a simple closed curve encircling the origin but not the branch points $\mathrm{e}^{\pm \mathrm{i}\theta}$, positively oriented. It is readily seen that C can be deformed into the bold paths, still denoted by C, as illustrated in Fig. 1. The integral in (1.5) seems closely related to Darboux's method, as has been dealt with in a preceding paper [13]. But this time it is of more interest; with the appearance of $h(\theta)$, along with the coalescing of a pair of branch points $\mathrm{e}^{\pm \mathrm{i}\theta}$ on the circle of convergence as $\theta \to 0$, there is a singularity in the exponent since $h(\theta) \sim \frac{a+b}{2\theta}$. Treating problems of this type is also of interest to us.

In view of the reflection formula [7, p. 7]

$$P_n(x; a, b) = (-1)^n P_n(-x; a, -b), \tag{1.6}$$