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The ABC of hyper recursions

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Dedicated to Roderick Wong on the occasion of his 60th birthday

Abstract

Each member of the family of Gauss hypergeometric functions

$$f_n = {}_2F_1(a+\varepsilon_1 n, b+\varepsilon_2 n; c+\varepsilon_3 n; z),$$

where a, b, c and z do not depend on n, and $\varepsilon_j = 0, \pm 1$ (not all ε_j equal to zero) satisfies a second order linear difference equation of the form

$$A_n f_{n-1} + B_n f_n + C_n f_{n+1} = 0.$$

Because of symmetry relations and functional relations for the Gauss functions, the set of 26 cases (for different ε_j values) can be reduced to a set of 5 basic forms of difference equations. In this paper the coefficients A_n, B_n and C_n of these basic forms are given. In addition, domains in the complex z-plane are given where a pair of minimal and dominant solutions of the difference equation have to be identified. The determination of such a pair asks for a detailed study of the asymptotic properties of the Gauss functions f_n for large values of n, and of other Gauss functions outside this group. This will be done in a later paper.

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1. Introduction

The Gauss hypergeometric functions

$$f_n = {}_2F_1\left(\begin{matrix} a+\varepsilon_1 n, b+\varepsilon_2 n \\ c+\varepsilon_3 n \end{matrix}; z\right) \tag{1.1}$$

where ε_j are integers, a, b, c and z do not depend on n, and

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} z^n, \quad |z| < 1 \tag{1.2}$$

satisfy a *second order linear difference equation* (also called a *three-term recurrence relation*) of the form

$$A_n f_{n-1} + B_n f_n + C_n f_{n+1} = 0. \tag{1.3}$$

For example, we have

$$(c-a)\,{}_2F_1\left(\begin{matrix} a-1, b \\ c \end{matrix}; z\right) + (2a + -c - z(a-b))\,{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) + a(z-1)\,{}_2F_1\left(\begin{matrix} a+1, b \\ c \end{matrix}; z\right) = 0, \tag{1.4}$$

in which we can replace a with $a+n$. Other examples are given in [1, p. 558].

In this paper we consider the 26 recursion relations with respect to n for the cases

$$\varepsilon_j = 0, \pm 1, \quad j = 1, 2, 3, \quad \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 \neq 0, \tag{1.5}$$

and by using symmetry relations and functional relations for the Gauss functions we assign a set of 5 basic forms from which the remaining 21 cases can be obtained.

A solution f_n of the recurrence relation (1.3) is said to be *minimal* if there exists a linearly independent solution g_n, of the same recurrence relation such that $f_n/g_n \to 0$ as $n \to \infty$. In that case g_n is called a *dominant* solution. When a recurrence admits a minimal solution (unique up to a constant factor), this solution should be included in any numerically satisfactory pair of solutions of the recurrence. Given a solution of the recurrence relation, it is crucial to know the character of the solution (minimal, dominant or none of them) in order to apply the recurrence relation in a numerically stable way. Indeed, if f_n is minimal as $n \to +\infty$, forward recurrence (increasing n) is an ill conditioned process because small initial errors will generally dominate the recursive solution by introducing an initially small component of a dominant solution; backward recurrence is well conditioned in this case. The opposite situation takes place for dominant solutions.

For each basic form we give the coefficients A_n, B_n and C_n, and after computing limits of the ratios $\beta = \lim_{n\to\infty}(A_n/C_n)$ and $\alpha = \lim_{n\to\infty}(B_n/C_n)$ we determine the zeros t_1 and t_2 of the characteristic polynomial $t^2 + \alpha t + \beta$, and we give curves in the z-plane where $|t_1| = |t_2|$. These curves enclose domains where a pair $\{f_n, g_n\}$ of minimal and dominant solutions of the difference equation has to be identified. For each basic form, and for each domain in the z-plane defined by the boundary curves belonging to that form, we give a number of candidates of minimal and dominant solutions. The selection of a suitable pair $\{f_n, g_n\}$ of minimal and dominant solutions can be done after a detailed study of the asymptotic