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Orthogonal confluent hypergeometric lattice polynomials

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Abstract

Here is a method of solving the difference-differential equations of the confluent hypergeometric differential equation using a generalized Pochhammer matrix product. This method provides a convenient analytical way to relate various solutions of the confluent hypergeometric function to each other when their parameters fall on the same point lattice. These solutions also are of interest to the general classification of orthogonal polynomials and the metrics used to generate them. This method generates Laurent polynomials over the complex domain that are an orthogonal system utilizing a 2×2 matrix weight function where the weight matrix has elements that are products of a Kummer solution and its derivative. The index-incremented Pochhammer matrix polynomials obey a 4×4 system of differential equations with a Frobenius solution involving non-commuting matrices that also extends these results to non-integer values but with infinite Laurent series. The termination condition for a polynomial series in the midst of infinite series sheds light on solving general systems of regular linear differential equations. The differential equations generalize Heun's double confluent equation with matrix coefficients. For a radiative transfer flux integral there is a distinct advantage of using these lattice polynomials compared to an asymptotic series/power series combination. We conjecture similar convergence properties for evaluations of confluent hypergeometric functions of either kind and that these matrix methods can be extended to gauss hypergeometric functions and generalized hypergeometric functions.

Keywords: Confluent hypergeometric functions; Independent coalescence; Orthogonal matrix polynomials; Heun's equation; Matrix differential equations; Pochhammer matrix product; Radiative transfer; Parametric function representation

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1. Introduction

Confluent hypergeometric functions have been utilized by mathematicians and physicists for centuries and the common basis of these two parameter functions was published by Kummer in 1836 [28]. Some of these functions were popularized in [28,33,6,14,21,18], the Bateman manuscripts, [3], and more recently by Luke and also by Slater in the Handbook of Mathematical Functions [HMF] [21,1], where a complex parameter case of coulomb wave functions is also described. Refs. [22,23] has provided a Lie group analysis of this equation. Raising and lowering operators and related factorization techniques have been developed for operators of the hypergeometric type by several groups [19,20]. Today we might say any family of functions that warp into each other as the two complex parameters a and b change in a finite region of the two complex planes are homotopic over that region and any set of homotopic functions whose parameters differ by integers are said to be on the same point lattice of the homotopy space and are said to be associated functions. The second order ordinary differential equation (ODE) also demands that for each homotopic solution there is a second solution, also homotopic. These solutions have been standardized as $M(a, b, z)$ or ${}_1F_1(a; b; z)$ and $U(a, b, z)$, the latter being necessary for completeness with positive integer indices of the parameter b where logarithmic singularities at the origin occur. The two kinds are bound by a Wronskian with an algebraic value that on particular occasions vanishes, but Kummer relationships fill this void and permit M and U to represent all solutions.

Slater's seminal book starts with the power series definition for contiguous M and obtains the six nearest neighbor differential-difference equations. These give rise to the algebraic recursion equations required to extend the included tables. The power series normalization requires M to be unity at the origin and U to be a pure power at infinity. This normalization of M has the disadvantage that it induces poles in b, the second parameter, when b takes on negative integer values and causes M to be undefined (HMF Eq. (13.1.1)).Otherwise $M(a, b, z)$ is holomorphic in the entire complex plane with respect to the parameters a and b and is so everywhere except at the origin for z. MacRobert's functions address this problem of regularization in parameters a and b [3], as does the normalization-free differential-equation based lattice polynomials described here.

Slater chose the unit cell for (a, b) of $[-1, 1] \times [0, 1]$ for the primary function in constructing the HMF tables for Chapter 13 rather than utilizing $[0, 1] \times [0, 1]$ on which the functions and its derivatives is specified. Slater's extension along with the Kummer relationships, allows the use of recursion relationships to extend the tables indefinitely except at isolated poles and except possibly for mathematical problems in evaluation. The recursion relationship has been the basis of computations implementations of U by Temme and others [32]. Muller [25] has given an overview of many other methods of evaluating M.

The solutions fall into pure-polynomial [30,10] and non-polynomial types. The polynomial solutions that may or may not be multiplied by a power or exponential weight include Laguerre and Hermite, as well as the non-classical associated Taylor polynomials, utilizing a finite number of terms in a Taylor series, associated with the incomplete gamma function $e^x \Gamma(n, x)$ and associated with exponential integrals $\mathrm{E}_n(x)$. These polynomials are not those that are the topic of this paper.

Section 2 deals with the operator approach that gives rise to the Pochhammer operator relationships and the Pochhammer matrices that are the foundation results of this presentation.

Section 3 establishes the orthogonality conditions for Kummer functions; this relationship corresponds to an apparently new type of scalar product space for them.

Orthogonal polynomials have wide uses ranging from statistical distributions, radiative transfer, to solutions of special problems in quantum mechanics, and are a key to establishing solution techniques of