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Plane wave decomposition in the unit disc: Convergence estimates and computational aspects

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Abstract

This paper deals with the numerical simulation of time-harmonic wave fields using progressive plane waves. It is shown that a plane wave travelling in arbitrary direction can be numerically recovered with an accuracy of the order of the machine precision with a collocation formulation and the square root of the machine precision with a least-square formulation. However, strongly evanescent and nearly singular wave fields cannot be properly recovered with standard double-precision floating-point arithmetic. Some of the ideas are applied to the elastic wave equation and a simple optimization algorithm is proposed to find a good compromise between the accuracy and the number of plane waves.

Keywords: Plane wave basis; Helmholtz equation; Elastodynamics; Finite precision computation

1. Introduction

Methods using superposition of progressive plane waves for the numerical simulation of time-harmonic wave problems generally falls in the much wider class of methods called Trefftz-type methods in which an approximate solution of a boundary value problem is built from the sets of functions that satisfy exactly the differential equation. These plane wave methods have been mainly developed for domain discretization schemes. Although this is not the place for a complete survey, one can cite the Ultra Weak Formulation introduced by Després for the Helmholtz equation [7,6,9] and recently extended for the elastodynamic equation [8] or the least-squares Trefftz-type elements [11]. Use of plane waves is also

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advocated in the Partition of Unity Method introduced by Babuška and Melenk [2] and applications for scattering problems can be found in [10,13,12]. All these techniques showed considerable improvements both in terms of degree of freedom reduction and accuracy compared with conventional discretization schemes. However, the question of numerical stability of the plane wave basis due to the poor conditioning of the resulting algebraic system remains an open problem. Sometimes described as basis ‘badness’ in quantum mechanics [3], this can bring severe limitations to the method if the wave field to be approximated is strongly evanescent. Though evanescent waves can theoretically be expressed as the singular limit of an angular superposition of real (i.e. progressive) plane waves [4], their associated coefficients become exponentially large so that only many-decimal arithmetic computation can recover the exact solution.

The present paper aims at bringing some new contributions to the understanding of these matters. Focusing on the Helmholtz equation in the unit disc, precise estimates for the plane wave basis approximation error (in the maximum-norm) as well as the conditioning number arising from both least square and collocation formulations are given in Section 2. In Section 3, some of the ideas developed for the Helmholtz problem are applied to the elastic wave equation.

2. Helmholtz equation

In this section, we consider the Helmholtz equation on a circular domain of diameter h. Without lack of generality we restrict ourselves to the particular case where the domain Ω is the unit disc by introducing the reduced wave number $\kappa = \pi h/\lambda$ (λ is the wavelength) so that the Dirichlet problem can be written as

$$\Delta u + \kappa^2 u = 0 \quad \text{on } \Omega, \tag{1}$$

$$u = g \quad \text{on } \gamma = \partial\Omega. \tag{2}$$

In the sequel, we call $\boldsymbol{x} = (x_1, x_2)$ the cartesian coordinates and (r, θ), its polar representation. We note $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_{L^2(\gamma)}$ the usual inner product and its associated norm of the Hilbert space $L^2(\gamma)$.

2.1. Error analysis

We assume that the boundary data g are given via its Fourier series as

$$g(\theta) = \sum_{n\in\mathbb{Z}} \hat{g}_n \mathrm{e}^{\mathrm{i}n\theta}, \tag{3}$$

where the series converges pointwise on $[0, 2\pi]$. Provided that the wave number κ is such that $J_n(\kappa) \neq 0$ for any integer $|n| < \kappa$, the unique solution is given by the infinite sum

$$u(\boldsymbol{x}) = \sum_{n\in\mathbb{Z}} \hat{g}_n \frac{J_n(\kappa r)}{J_n(\kappa)} \mathrm{e}^{\mathrm{i}n\theta}. \tag{4}$$

We define by u_N the truncated sum (4) up to the order N and we call

$$\Psi(\kappa; \phi, \boldsymbol{x}) = \exp(\mathrm{i}\kappa(x_1 \cos\phi + x_2 \sin\phi)) \tag{5}$$